Math 507, Midterm
November 8, 2004

Problem 1. Let $\mathbf{v}_1 = (1,1,1,1,1,1,1)$, $\mathbf{v}_2 = (1,0,1,0,1,0,1)$, $\mathbf{v}_3 = (1,1,1,1,2,1,1)$, $\mathbf{v}_4 = (0,1,0,0,0,1,0)$, $\mathbf{v}_5 = (1,1,0,0,0,1,1)$ and $U = \text{span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\})$. Let $W \subseteq K^7$ be the space of solutions to the system $x_3 - x_5 = 0$, $x_4 = 0$.

a) Find a homogeneous system of equations with the space of solutions equal to U. **(6 points)**

b) Find a basis of $U \cap W$. **(6 points)**

c) What can you say about $U + W$? **(4 points)**

Problem 2. a) Find the reduced row-echelon form of the matrix $A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \end{pmatrix}$. **(6 points)**

b) Express A as a product of elementary matrices. **(5 points)**

c) Find the determinant of A. **(5 points)**

d) Write down the matrix $A_{4,4}$ and compute its inverse. Verify your answer. **(5 points)**

Problem 3. a) A linear transformation $S : \mathbb{R}^6 \longrightarrow \mathbb{R}^4$ is given by the matrix $A = \begin{pmatrix} 2 & 3 & 1 & 4 & -9 & 17 \\ 1 & 1 & 1 & 1 & -3 & 6 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{pmatrix}$

Find bases of the kernel and of the image of S. **(6 points)**

b) The matrix of a linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ in the ordered basis $(2,1,1), (2,2,1), (3,2,2)$ of $\mathbb{R}^3$ and the ordered basis $(2,1,0,0), (0,0,1,1), (0,1,0,1), (1,0,1,0)$ of $\mathbb{R}^4$ equals $B = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$.

Find the matrix of T in the standard bases. **(6 points)**

Problem 4. Answer true or false. In each case provide an explanation **(4 points each)**.

a) There exists a surjective linear transformation $T : \mathbb{R}^7 \longrightarrow \mathbb{R}^3$ whose kernel has dimension 5.

b) If $T : V \longrightarrow V$ is a linear transformation then $\text{Im}T$ is a $T-$invariant subspace.

c) The function $T(a,b) = (a^2, ab, b^2)$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$.

d) There exists a linear transformation $T : K^5 \longrightarrow K^5$ such that $\ker T = \text{Im}T$.

e) $\det(A+B) = \det A + \det B$ for any two 3×3 matrices A, B.

f) The matrices $\begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & -1 & -2 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{pmatrix}$ are similar.

Problem 5. a) Let $T : V \longrightarrow V$ be a linear transformation. Suppose that every one dimensional subspace of V is T-invariant. Prove that $T = aI$ for some constant a. **(9 points)**

b) Let $T : V \longrightarrow V$ be a linear transformation. Suppose that the annihilator of a vector $u \in V$ is $p_u = x + 1$ and the annihilator of v is $p_v = x - 1$. Prove that the annihilator of $u + v$ equals $x^2 - 1$. **(9 points)**

c) Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation such that $T(v) - v \in \text{Im}(T)^{\perp}$ for all $v \in \mathbb{R}^n$. Prove that $T^2 = T$. **(9 points)**

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The following problems are optional. You may earn extra points, but work on these problems only after you are done with the other problems.

Problem 6. Let A be an invertible matrix with all entries integers. Show that $\det A$ is an integer. Prove that all entries of A^{-1} are integers iff $\det A = \pm 1$. **(8 points)**

Problem 7. Let A be a 4×4 matrix whose all entries are from the set $\{-3, 2\}$. Prove that $\det A$ is divisible by 125. **(8 points)**

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$$\begin{pmatrix} 2 & 3 & 1 & 4 & -9 & 17 \\ 1 & 1 & 1 & 1 & -3 & 6 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 & 0 & -2 & 3 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 & -2 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{pmatrix}$$