Solutions to the Midterm

Solution to Problem 1. Let $\mathbf{v}_1 = (1,1,1,1,1,1,1)$, $\mathbf{v}_2 = (1,0,1,0,1,0,1)$, $\mathbf{v}_3 = (1,1,1,1,2,1,1)$, $\mathbf{v}_4 = (0,1,0,0,0,1,0)$, $\mathbf{v}_5 = (1,1,0,0,0,1,1)$ and $U = \mathrm{span}(\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4,\mathbf{v}_5\})$. Let W be the space of solutions to the system $x_3 - x_5 = 0$, $x_4 = 0$. Both U and W are subspaces of K^7.

a) In order to find a homogeneous system of equations with the space of solutions equal to U we determine a basis of $U^\perp$, i.e. we find a basis of solutions to the homogeneous system of equations with coefficient matrix equal to

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

The reduced row echelon form of this matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

We see that there are two non-pivot columns which correspond to free variables x_6 and x_7. Thus $U^\perp$ has dimension 2 and basis $(0,-1,0,0,0,1,0)$, $(-1,0,0,0,0,0,1)$. Consequently, U is the solution space to the system of 2 equations:

$$-x_2 + x_6 = 0, \quad -x_1 + x_7 = 0.$$

Note also that $\dim U = 7 - \dim U^\perp = 5$.

b) We found in a) equations for U and we are given equations for W. The intersection $U \cap W$ is the solution space to the combined system, i.e. to the system of equations

$$-x_2 + x_6 = 0, \quad -x_1 + x_7 = 0, \quad x_3 - x_5 = 0, \quad x_4 = 0$$

The coefficient matrix of this system equals

$$\begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

and its reduced row-echelon form is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

We have 3 free variables and a basis of solutions $(0,0,1,0,1,0,0)$, $(0,1,0,0,0,1,0)$, $(1,0,0,0,0,0,1)$. This is a basis of $U \cap W$ and therefore $U \cap W$ has dimension 3.

c) Note that $W = \{(0,0,1,0,-1,0,0),(0,0,0,1,0,0,0)\}^\perp$. Since the vectors $(0,0,1,0,-1,0,0)$, $(0,0,0,1,0,0,0)$ are linearly independent, we have

$$\dim W = 7 - \dim W^\perp = 7 - 2 = 5.$$

Thus

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W) = 5 + 5 - 3 = 7.$$

Since $U + W$ is a subspace of K^7, we have $U + W = K^7$.

Solution to Problem 2. Let $A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \end{pmatrix}$.

a) In order to find the reduced row-echelon form of A perform the following elementary row operations:

$$T_{1,2}, E_{3,1}(-2), E_{4,1}(-1), E_{1,2}(-2), E_{3,2}(4), E_{4,2}(1), S_3(-1), E_{1,3}(-1), E_{2,3}(-1), E_{4,3}(2), S_4(1/5)$$

We see that the reduced row-echelon form of A is the identity matrix. Thus

$$S_4(1/5)E_{4,3}(2)E_{2,3}(-1)E_{1,3}(-1)S_3(-1)E_{4,2}(1)E_{3,2}(4)E_{1,2}(-2)E_{4,1}(-1)E_{3,1}(-2)T_{1,2}A = I$$

b) From a) we get that

$$A = (S_4(1/5)E_{4,3}(2)E_{2,3}(-1)E_{1,3}(-1)S_3(-1)E_{4,2}(1)E_{3,2}(4)E_{1,2}(-2)E_{4,1}(-1)E_{3,1}(-2)T_{1,2})^{-1} =$$

$$= T_{1,2}^{-1}E_{3,1}(-2)^{-1}E_{4,1}(-1)^{-1}E_{1,2}(-2)^{-1}E_{3,2}(4)^{-1}E_{4,2}(1)^{-1}S_3(-1)^{-1}E_{1,3}(-1)^{-1}E_{2,3}(-1)^{-1}E_{4,3}(2)^{-1}S_4(1/5)^{-1} =$$

$$= T_{1,2}E_{3,1}(2)E_{4,1}(1)E_{1,2}(2)E_{3,2}(-4)E_{4,2}(-1)S_3(-1)E_{1,3}(1)E_{2,3}(1)E_{4,3}(-2)S_4(5)$$

c) Recall that $\det E_{i,j}(a) = 1$, $\det T_{i,j} = -1$ and $\det S_i(a) = a$. Using this and b) we see that $\det A = (-1) \cdot (-1) \cdot 5 = 5$.

d) In general, $A_{s,t}$ is the matrix obtained from A by removal of s-th row and t-th column. Thus

$$A_{4,4} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 0 & 1 \end{pmatrix}.$$

In order to find $(A_{4,4})^{-1}$ we row-reduce the matrix

$$\left(\begin{array}{ccc:ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

and get

$$\left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & 5 & -2 & 1 \\ 0 & 0 & 1 & -4 & 2 & -1 \end{array}\right).$$

Thus

$$A_{4,4}^{-1} = \begin{pmatrix} 2 & -1 & 1 \\ 5 & -2 & 1 \\ -4 & 2 & -1 \end{pmatrix}.$$

In order to verify the answer we perform the multiplication

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ 5 & -2 & 1 \\ -4 & 2 & -1 \end{pmatrix} = I.$$

Solution to Problem 3. a) A linear transformation $S : \mathbb{R}^6 \longrightarrow \mathbb{R}^4$ is given by the matrix $A = \begin{pmatrix} 2 & 3 & 1 & 4 & -9 & 17 \\ 1 & 1 & 1 & 1 & -3 & 6 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{pmatrix}$ In order to find bases of the kernel and of the image of S we find the reduced row-echelon form

$$\begin{pmatrix} 1 & 0 & 2 & 0 & -2 & 3 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

of the matrix A. Recall that ImS is the column space of A so the pivot columns of A form a basis of ImS. Thus $(2,1,1,2)$, $(3,1,1,2)$, $(4,1,2,3)$ is a basis of ImS.

The kernel $\ker S$ is the solution space to the homogeneous system of linear equations with coefficient matrix A. Thus from the reduced row-echelon form of A we deduce that $(-2,1,1,0,0,0)$, $(2,-1,0,2,1,0)$, $(-3,-1,0,-2,0,1)$ is a basis of $\ker S$.

b) The matrix of a linear transformation $T:\mathbb{R}^3 \longrightarrow \mathbb{R}^4$ in the ordered basis $\mathbf{v}: (2,1,1),(2,2,1),(3,2,2)$ of $\mathbb{R}^3$ and the ordered basis $\mathbf{w}: (2,1,0,0),(0,0,1,1),(0,1,0,1),(1,0,1,0)$ of $\mathbb{R}^4$ equals $B = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$.

In other words, $B = M^{\mathbf{w}}_{\mathbf{v}}(T)$. The matrix $M^{\mathbf{e}}_{\mathbf{e}}(T)$ of T in the standard bases is given by

$$M^{\mathbf{e}}_{\mathbf{e}}(T) = M^{\mathbf{e}}_{\mathbf{w}}(I) M^{\mathbf{w}}_{\mathbf{v}}(T) M^{\mathbf{v}}_{\mathbf{e}}(I).$$

We have

$$M^{\mathbf{e}}_{\mathbf{w}}(I) = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

and

$$M^{\mathbf{e}}_{\mathbf{v}}(I) = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix}.$$

Thus

$$M^{\mathbf{v}}_{\mathbf{e}}(I) = M^{\mathbf{e}}_{\mathbf{v}}(I)^{-1} = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 & -2 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{pmatrix}$$

It follows that

$$M^{\mathbf{e}}_{\mathbf{e}}(T) = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 & -2 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 2 & -10 \\ 3 & 1 & -4 \\ 2 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}.$$

Solution to Problem 4.

a) False. A surjective linear transformation $T:\mathbb{R}^7 \longrightarrow \mathbb{R}^3$ has kernel of dimension $\dim\ker T = 7 - \dim \mathrm{Im}T = 7-3 = 4 \neq 5$.

b) True. If $v \in \mathrm{Im}T$ then $T(v) \in \mathrm{Im}(T)$ by the very definition of the image. Thus ImT is a $T-$invariant subspace.

c) False. We have $T(2(1,1)) = T(2,2) = (4,4,4) \neq 2T(1,1) = (2,2,2)$ so T is not a linear transformation.

d) False. Let $a = \dim\ker T$, $b = \dim \mathrm{Im}T$, so $a+b=5$. If $a=b$, then $2a=5$, which is not possible.

e) False. There are many counterexamples. For example, take $A=I=B$. Then $\det(A+B) = \det 2I = 8$ and $\det A + \det B = 1+1 = 2$.

f) False. The matrix $\begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix}$ has trace 6 and the matrix $\begin{pmatrix} 2 & -1 & -2 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{pmatrix}$ has trace 5 so they are not similar.

Problem 5. a) Let $T:V\longrightarrow V$ be a linear transformation such that every one dimensional subspace of V is T-invariant. Let $v \in V$ be a non-zero vector. Since the one dimensional subspace span$\{v\}$ is T invariant, $T(v) = a(v)v$ for some scalar $a(v)$. We claim that $a(v)$ must be the same for all vectors v. In fact, let w be another non-zero vector. If $w \in \mathrm{span}\{v\}$ then $w = cv$ for some scalar c, so

$T(w) = T(cv) = cT(v) = ca(v)v = a(v)(cv) = a(v)w$, so $a(w) = a(v)$. If $w \notin \text{span}\{v\}$, then v and w are linearly independent. Note that

$$T(v+w) = a(v+w)(v+w) = a(v+w)v + a(v+w)w.$$

On the other hand,

$$T(v+w) = T(v) + T(w) = a(v)v + a(w)w.$$

It follows that $a(v+w)v + a(v+w)w = a(v)v + a(w)w$, i.e. that

$$(a(v+w) - a(v))v + (a(v+w) - a(w))w = 0.$$

The linear independence of v and w implies that $a(v+w) - a(v) = 0 = a(v+w) - a(w)$, i.e. that $a(v) = a(v+w) = a(w)$. This proves our claim that $a(v) = a$ does not depend on v. Thus $T(v) = av$ for all $v \in V$, i.e. $T = aI$.

b) Let $T : V \longrightarrow V$ be a linear transformation. Suppose that the annihilator of a vector $u \in V$ is $p_u = x+1$ and the annihilator of v is $p_v = x-1$. This means that $(T+I)(u) = 0$ and $(T-I)(v) = 0$, i.e. $T(u) = -u$ and $T(v) = v$. Furthermore, $u \neq 0$ and $v \neq 0$ (since $< 0 >= \{0\}$ has dimension 0 and both $< u >$ and $< v >$ have dimension 1).

The vectors $u+v$ and $T(u+v) = -u+v$ are linearly independent. In fact, suppose that $a(u+v) + b(-u+v) = 0$, i.e. $(b-a)u = (b+a)v$ for some scalars a, b. Applying T to the last equality yields

$$(a-b)u = T((b-a)u) = T((b+a)w) = (b+a)w.$$

It follows that $(a-b)u = (b-a)u$. Since $u \neq 0$, we conclude that $a = b$ and $0 = 2aw$. Thus $2a = 0$, since $w \neq 0$. We see that $a = 0 = b$ (we must assume that $2 \neq 0$, i.e. that the field of scalars is not of characteristic 2).

We proved that $u+v$ and $T(u+v) = -u+v$ are linearly independent. But $T^2(u+v) = (u+v) = 1 \cdot (u+v) + 0 \cdot T(u+v)$ so the annihilator of $u+v$ is indeed $x^2 - 1$.

c) Let U be a subspace of $\mathbb{R}^n$. Suppose that $u = (u_1, ..., u_n) \in U \cap U^{\perp}$. Then $u \cdot u = 0$. But $u \cdot u = u_1^2 + u_2^2 + ... + u_n^2 = 0$ iff $u_1 = u_2 = ... = u_n = 0$ (here we use the fact that our field is $\mathbb{R}$, so that sum of squares can be zero if and only if each summand is 0; this is not true for complex numbers or finite fields). We see that $u = 0$, i.e. $U \cap U^{\perp} = \{0\}$.

We apply the above observation to $U = \text{Im}(T)^{\perp}$, so $\text{Im}(T)^{\perp} \cap \text{Im}(T) = \{0\}$. Let $v \in \mathbb{R}^n$. Then $T^2(v) - T(v) = T(T(v) - v) \in \text{Im}(T)$. On the other hand, $T^2(v) - T(v) = T(u) - u \in \text{Im}(T)^{\perp}$, where $u = T(v)$. It follows that $T^2(v) - T(v) \in \text{Im}(T)^{\perp} \cap \text{Im}(T) = \{0\}$., i.e. $T^2(v) = T(v)$. Thus $T^2 = T$.

Problem 6. Let $A = (a_{i,j}$ be an invertible $n \times n$ matrix with all entries integers. Recall that

$$\det A = \sum_{\tau} \text{sign}(\tau) \Pi_{i=1}^{n} a_{i,\tau(i)}.$$

It is clear now that $\det A$ is an integer (alternatively, use induction on n and row (column) expansion). If all entries of A^{-1} are integers then $\det A^{-1} = 1/\det A$ is an integer. Thus both $\det A$ and $1/\det A$ are integers. It follows that $\det A = \pm 1$.

Suppose now that $\det A = \pm 1$. Recall that $A^{-1} = (\det A)^{-1} A^D$, where $A^D = (d_{i,j})$ is the $n \times n$ matrix such that $d_{i,j} = (-1)^{i+j} \det(A_{j,i})$. Thus if A has integral entries then so does A^D. Since in our case $A^{-1} = \pm A^D$, the matrix A^{-1} has integral entries.

Problem 7. Let A be a 4×4 matrix whose all entries are from the set $\{-3, 2\}$. Apply the elementary row operations $E_{1,4}(-1)$, $E_{2,4}(-1)$, $E_{3,4}(-1)$ to A. The resulting matrix B has the same determinant as A. Note that the all entries in the first three rows of B are in $\{\pm 5, 0\}$. It follows that $S_1(1/5)S_2(1/5)S_3(1/5)B$ has integral entries. Thus $\det(S_1(1/5)S_2(1/5)S_3(1/5)B) = \det B/125$ is an integer. In other words, 125 divides $\det B = \det A$.