CONSTRUCTING GROUPS

There are several methods of building new groups out of already constructed. We are going to described some of them.

Direct Products.

Let G_i, $i \in I$ be a collection of groups indexed by a set I. The **direct product** $G = \Pi_{i\in I} G_i$ is the set of all functions $f : I \longrightarrow \bigcup_{i\in I} G_i$ such that $f(i) \in G_i$ for all $i \in I$ (i.e. it is the product of the sets G_i) equipped with group operation $\bullet$ defined by $(f \bullet g)(i) = f(i)g(i)$ (multiplication in the group G_i) for all $i \in I$. The verification that G is a group with respect to this operation is starighforward. Each G_i can be identified with a subgroup of G as follows: $g \in G_i$ corresponds to the function h such that $h(i) = g$ and $h(j) = e$ (the identity in G_j) for all $j \neq i$. It is easy to see that the groups G_i are pairwise disjoint, normal subgroups of G and elemnts in G_i commute with elements in G_j for $i \neq j$. For each i there is a natural homomorphism $\pi_i : G \longrightarrow G_i$ (called **projection** onto G_i) defined by $\pi_i(f) = f(i)$. The direct product has the following property:

For any family of group homomorphisms $f_i : H \longrightarrow G_i$, $i \in I$, there is unique homomorphism $f : H \longrightarrow G$ such that $\pi_i f = f_i$ for all $i \in I$. It is defined by $f(h)(i) = f_i(h)$ for all $i \in I$.

From the point of view of general category theory, the direct product is just the product in the category of groups.

Direct sums.

The **direct sum** of a family of groups G_i, $i \in I$, is the subgroup $D = \sum_{i\in I} G_i$ of $\Pi_{i\in I} G_i$ which consists of all functions which satisfy $f(i) = e$ (the unit in G_i) for all but a finite number of $i \in I$. It is easy to see that the direct sum is a normal subgroup of the direct product. Moreover, D contains all the subgroups G_i and these subgroups generate D. The direct sum has the following universal property:

For any family of group homomorphisms $g_i : G_i \longrightarrow H$, $i \in I$, such that $[g_i(G_i), g_j(G_j)] = \{e\}$ for all pairs $i \neq j$ there is unique homomorphism $g : \sum_{i\in I} G_i \longrightarrow H$ which coincides with g_i on G_i for all $i \in I$. For any $f \in \sum_{i\in I} G_i$ the image $g(f)$ is defined as the product of all $g_i(f(i))$ for $i \in I$. Note that this is well defined, since the product involves only a finite number of nonidentity elements in H and the order in which this product is taken is irrelevant due to the fact that the multiplied elements pairwise commute.

In particular, if all G_i are commutative then $\sum_{i\in I} G_i$ is commutative and it is the coproduct of G_i in the category of *commutative* groups.

The direct sum of a finite family of groups coincides with the direct product of this family (and for infinite families of nontrivial groups these two groups never coincide).

We are going to discuss now a direct product of a finite family of groups G_1,... G_n more carefully. In this case $G = \Pi_{i=1}^n G_i = G_1 \times ... \times G_n$ can be identified with $n-$tuples $(g_1, ..., g_n)$ such that $g_i \in G_i$ and the multiplication is performed coordinatewise. We have seen that G_i are normal subgroups of G, they pairwise commute, any two have trivial intersection and they generate G. In fact more is true: the kernel $\ker \pi_i$ coincides with the product of all G_j with $j \neq i$ (the order in which the product is taken does not matter since the G_i's pairwise commute), so $\ker \pi_i \cap G_i = \{e\}$ for all i. This properties in fact characterize finite direct products:

Proposition 1. *Suppose G is a group with normal subgroups G_1,...,G_n such that:*

- *G is generated by $G_1 \cup ... \cup G_n$;*
- *for every i, the product of all G_j with $j \neq i$ has trivial intersection with G_i.*

Then $G = G_1...G_n$ and it is naturally isomorphic with $G_1 \times ... \times G_n$.

The proof follows easily from the universal property of direct sums once the pairwise commutativity of G_i and G_j is established for all $i \neq j$. This in turn follows from the following simple but fundamental observation:

Lemma 1. *If K, L are normal subgroups of a group G such that $K \cap L = \{e\}$ then $[K, L] = \{e\}$.*

In fact, if $k \in K$ and $l \in L$ then $[k,l] = (klk^{-1})l^{-1} = k(lk^{-1}l^{-1})$, so $[k,l] \in K \cap L$, hence it is trivial.

Exercise. Let G be the direct product of countable many copies of $\mathbb{Z}$, i.e. $G = \Pi_{i=1}^{\infty} A_i$, where $A_i = \mathbb{Z}$ for all i. Let H be the direct sum of these groups.

a) Prove that if $\phi : G \longrightarrow \mathbb{Z}$ is a homomorphism such that $H < \ker \phi$, then $\ker \phi = G$.

b) Prove that G is not isomorphic to a direct sum of the form $\sum_{i \in I} \mathbb{Z}$.

c) An abelian group without elements of finite order B is called **slender**, if every homomorphisms $\psi : G \longrightarrow B$ maps all but a finite number of A_i to the identity of B. Prove that $\mathbb{Z}$ is slender.

d) Prove that there is no epimorphism of G onto H.

Exercise. Let G be a finite group and H a minimal nontrivial normal subgroup of G (i.e. nontrivial normal subgroup of G which does not contain any proper, nontrivial normal subgroup of G). Prove that H is a direct product of several copies of a simple group.

Semidirect products.

As we have seen above, if K, H are normal subgroups of G such that $K \cap H = \{e\}$ and $K \cup H$ generate G then G is isomorphic to a direct product $K \times H$.

Suppose now that we keep all the assumptions except that we no longer require H to be a normal subgroup. Then still $G = KH$ and each element of G can be uniquely written as kh with $k \in K$ and $h \in H$. Thus, as sets G and $K \times H$ can be identified. But it is no longer true that multiplication in $K \times H$ obtained from this identification is coordinatewise, i.e. it is in general not true that $(kh)(k'h') = (kk')(hh')$. Instead, we have the following

$$(kh)(k'h') = (k(hk'h^{-1}))(hh').$$

In other words, in order to recover G we not only need to know K and H but also the way in which elements of H interact with elements of K. To spell this out more precisely we need to discuss groups of automorphisms of groups.

For any group G, the set of all automorphisms of G forms a group under composition which is usually denoted by $\mathrm{Aut}G$. Given G, it is usually a challenging problem to understand $\mathrm{Aut}G$. Note that for every $g \in G$ conjugation by g is an automorphism of G, since $g(ab)g^{-1} = (gag^{-1})(gbg^{-1})$, and the inverse automorphism is conjugation by g^{-1}. It is straightforward to verify that the function $\mathrm{con} : G \longrightarrow \mathrm{Aut}G$ which takes g to conjugation by g is a group homomorphism. The image of con is denoted by $\mathrm{Inn}G$ and its elements are called **inner automorphisms**. The kernel of con coincides with the center of G, so if G has trivial center, it can be considered as a subgroup of $\mathrm{Aut}G$.

More generally, if N is a normal subgroup of G then for any $g \in G$, conjugation by g induces an automorphism of N and we get in this way a group homomorphism from G to $\mathrm{Aut}N$. The kernel of this homomorphism is called the **centralizer** of N in G.

Exercise. Show that $\mathrm{Inn}G$ is a normal subgroup of $\mathrm{Aut}G$. The quotient $\mathrm{Aut}G/\mathrm{Inn}G$ is usually denoted by $\mathrm{Out}G$ and called the group of **outer** isomorphisms of G.

Exercise. Describe all automorphisms of a cyclic group of order n.

Exercise. Prove that $\mathrm{Aut}S_n$ is isomorphic to S_n for all $n \neq 2, 6$. Show that for $n = 2, 6$ this is false. Describe $\mathrm{Aut}A_n$ for $n \geq 5$.

Let us return to our discussion. The way H and K interact is best described by the homomorphism $\phi : H \longrightarrow \mathrm{Aut}K$ which takes h to the automorphism of K given by conjugation by h. We can now write

$$(kh)(k'h') = (k\phi(h)(k'))(hh').$$

In other words, K, H and ϕ determine G.

This observation leads to the definition of a semidirect product of groups. Let K, H be groups and $\phi : H \longrightarrow \mathrm{Aut}K$ a group homomorphism. The **semidirect** product $K \rtimes_\phi H$ is the set $K \times H$ with multiplication $\bullet$ given by

$$(k, h) \bullet (k', h') = (k\phi(h)(k'), hh').$$

It is a simple exercise to verify that $\bullet$ is indeed a group operation. When there is no confusion about ϕ we usually write just $K \rtimes H$ for the semidirect product. An element of the form (k, e) can be identified with k and element of the form (e, h) can be identified with h. In this way K becomes a normal subgroup of $K \rtimes H$, H becomes a subgroup and $hkh^{-1} = \phi(h)(k)$ for all $h \in H$ and $k \in K$. Note that if ϕ is the trivial homomorphism, then the semidirect product coincides with the direct product.

Exercise. a) Prove that if $\phi, \psi : H \longrightarrow \mathrm{Aut}K$ are such that $\phi^{-1}\psi$ is an inner automorphism, then the groups $K \rtimes_\phi H$ and $K \rtimes_\psi H$ are isomorphic.

b) Suppose furthermore that there is no surjective map from K onto $\mathbb{Z}$ and that $H = \mathbb{Z}$. Show that $K \rtimes_\phi H$ and $K \rtimes_\psi H$ are isomorphic iff ϕ^ϵ and ψ are conjugate in $\mathrm{Out}K$, where $\epsilon = 1$ or $\epsilon = -1$.

Exercise. a) Show that D_n is isomorphic to $C_n \rtimes_\phi C_2$ for some ϕ, where C_n is the cyclic group of order n.

b) Prove that $D_\infty \approx \mathbb{Z} \rtimes_\phi C_2$ for some ϕ.

Exercise. Let $G = \oplus_{i=-\infty}^{\infty} C$, where $C = \{0, 1\}$ is the group of order 2. Thus G consists of all functions $f : \mathbb{Z} \longrightarrow C$ such that $f(i) = 0$ for all but finitely many i. Define an automorphism $t : G \longrightarrow G$ by $(tf)(i) = f(i-1)$ (so it is a shift). Let $\phi : \mathbb{Z} \longrightarrow \mathrm{Aut}G$ be given by $\phi(1) = t$. Set $H = G \rtimes \mathbb{Z}$.

a) Prove that H is generated by $(0, 1)$ and $(g, 0)$ where $g(0) = 1$ and $g(i) = 0$ for all $i \neq 0$.

b) Show that $[H, H]$ is the subgroup of G which consisits of all f for which $f(i) = 1$ for an even number of i.

c) Conclude that the derived group of a free group on two generators is not finitely generated (this is true for any nonabelian free group).

Free Products.

We have seen that the product in the category of groups coincides with the direct product. Is there a coproduct in the category of groups? Let G_i, $i \in I$ be a family of groups. The categorical coproduct is by definition a group G together with group homomorphisms $\tau_i : G_i \longrightarrow G$ such that for any group H and homomorphisms

$f_i : G_i \longrightarrow H$ there is unique homomorphism $f : G \longrightarrow H$ such that $f_i = f\tau_i$ for all i.

Coproducts indeed exist in the category of groups and they usually are called **free products**. To define them, let X be the disjoint union of the sets G_i, $i \in I$. Consider the free group $F(X)$. To avoid any confusion, we denote the multiplication in G_i by $\bullet_i$, and the unit element by e_i. Let R be the subset of $F(X)$ which consists of all words of the form $ab(a \bullet_i b)^{-1}$ where $i \in I$ and $a, b \in G_i$. The **free product** $\bigstar_{i\in I} G_i$ is defined as $< X|R >$, i.e. it is the quotient of $F(X)$ by the smallest normal subgroup which contains R.

Exercise. Verify that the free product defined above is indeed the coproduct in the category of groups.

Exercise. Show that the free product can be defined alternatively as follows. Consider the set of all words on X (notation as above). For $x \in X$ we write $i(x) = i$ if $x \in G_i$. We say that a word $w = a_1 ... a_k$ is proper if either it is the empty word or $a_j \neq e_{i(a_j)}$ for all $j = 1, 2, ..., k$ and $i(a_j) \neq i(a_{j+1})$ for $j = 1, ..., k-1$. We define multiplication of proper words by induction on the length of the words as follows

- $e \centerdot w = w \centerdot e = w$ for all proper words w;
- suppose that the multiplication of proper words of lengths $\leq k-1$ has been defined. For any proper words $w = y_1 y_2 ... y_l$ and $w' = y'_1 ... y'_m$ with $l, m \leq k$ define

$$w \centerdot w' = \begin{cases} y_1 ... y_l y'_1 ... y'_m & \text{if } i(y_l) \neq i(y'_1), \\ y_1 ... (y_l \bullet_i y'_1) ... y'_m & \text{if } i = i(y_l) = i(y'_1) \text{ and } y_l^{-1} \neq y'_1, \\ (y_1 ... y_{l-1}) \centerdot (y'_2 ... y'_m) & \text{if } y_l^{-1} = y'_1. \end{cases}$$

Verify that so defined product makes the set of reduced words a group and that this group is isomorphic to the free product $\bigstar_{i\in I} G_i$.

Exercise. a) Show that the free group $F(X)$ can be identified with $\bigstar_{i\in X} G_i$, where $G_i = \mathbb{Z}$ for all $i \in X$.

b) Prove that D_∞ is isomorphic to $C_2 \star C_2$ where C_2 is the group of order 2.

c) Prove that $PSL_2(\mathbb{Z})$ is isomorphic to $C_2 \star C_3$, where C_k is the cyclic group of order k.