Homework 2

Problem 1. Let G be a free group and $g \in G$ a nontrivial element. Prove that the centralizer of g in G is an infinite cyclic group.

Problem 2. Let G be a free group and H a normal subgroup of G of infinite index. Prove that H is not finitely generated.

Problem 3. Let G_i, $i \in I$ be a family of groups. Let X be the disjoint union of the sets G_i, $i \in I$. Consider the free group $F(X)$. To avoid any confusion, we denote the multiplication in G_i by $\bullet_i$, and the unit element by e_i. Let R be the subset of $F(X)$ which consists of all words of the form $ab(a \bullet_i b)^{-1}$ where $i \in I$ and $a, b \in G_i$. The **free product** $\bigstar_{i\in I} G_i$ is defined as the quotient of $F(X)$ by the smallest normal subgroup which contains R. For every $i \in I$ and $g \in G_i$ we define $\tau_i(g)$ as the image of g (considered as an element of $X \subseteq F(X)$) in $G = \bigstar_{i\in I} G_i$ (under the quotient map).

a) Prove that $\tau_i : G_i \longrightarrow G$ is a homomorphism.

b) Prove that for any group H and homomorphisms $f_i : G_i \longrightarrow H$ there is unique homomorphism $f : G \longrightarrow H$ such that $f_i = f\tau_i$ for all i. Conclude that τ_i are injective.

Remark. This result means that the free product is a coproduct in the category of groups.

c) Show that the free product is characterized by b), i.e. that if a group $\overline{G}$ and homomorphisms $\overline{\tau_i} : G_i \longrightarrow \overline{G}$ have the property described in b) then there is a unique isomorphism $\phi : G \longrightarrow \overline{G}$ such that $\phi\tau_i = \overline{\tau_i}$ for all i. Use this to prove that the free group $F(X)$ can be identified with $\bigstar_{i\in X} G_i$, where $G_i = \mathbb{Z}$ for all $i \in X$.

d)Show that the free product can be defined alternatively as follows. Consider the set of all words on X (notation as above). For $x \in X$ we write $i(x) = i$ if $x \in G_i$. We say that a word $w = a_1...a_k$ is proper if either it is the empty word or $a_j \neq e_{i(a_j)}$ for all $j = 1, 2, ..., k$ and $i(a_j) \neq i(a_{j+1})$ for $j = 1, ..., k-1$. We define multiplication of proper words by induction on the length of the words as follows

- $e \centerdot w = w \centerdot e = w$ for all proper words w;

- suppose that the multiplication of proper words of lengths $\leq k-1$ has been defined. For any proper words $w = y_1 y_2 ... y_l$ and $w' = y'_1 ... y'_m$ with $l, m \leq k$ define

$$w \centerdot w' = \begin{cases} y_1 ... y_l y'_1 ... y'_m & \text{if } i(y_l) \neq i(y'_1), \\ y_1 ... (y_l \bullet_i y'_1) ... y'_m & \text{if } i = i(y_l) = i(y'_1) \text{ and } y_l^{-1} \neq y'_1, \\ (y_1 ... y_{l-1}) \centerdot (y'_2 ... y'_m) & \text{if } y_l^{-1} = y'_1. \end{cases}$$

Verify that so defined product makes the set of proper words a group and that this group is isomorphic to the free product $\bigstar_{i \in I} G_i$.

Problem 4. a) Let G be a group acting on a set X, let G_1, G_2 be subgroups of G and let H be the subgroup generated by G_1 and G_2. Suppose that $|G_1| \geq 3$ and $|G_2| \geq 2$. Suppose that there are two nonempty subsets X_1, X_2 of X such that X_2 is not included in X_1, $g(X_2) \subseteq X_1$ for all $1 \neq g \in G_1$ and $g(X_1) \subseteq X_2$ for all $1 \neq g \in G_2$. Prove that H is isomorphic to the free product $G_1 * G_2$.

This result is often called the **Table-Tennis Lemma**, **Ping-Pong Lemma** or **Schottky Lemma**.

b) Consider the group $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm I\}$. It acts on the real line with added point at infinity $\mathbb{R} \cup \{\infty\}$ by fractional linear transformations, i.e.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} r = \frac{ar+b}{cr+d}$$

where the right hand side is a/c if $r = \infty$ and is ∞ if $r = -d/c$ (if $c = 0$, both a/c and $-d/c$ equal ∞). Let $q \geq 3$ be an integer and $t = 2\cos(\pi/q)$. Define matrices A, B as follows

$$A = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Prove that the images of A, B in $PSL_2(\mathbb{R})$ generate a subgroup Γ_t isomorphic to $C_q \star C_2$, where C_k is the cyclic group of order k (note that AB has order q).

Remark. The groups Γ_t are called Hecke groups.