Homework 3

Problem 1. a) For any integer $r \geq 0$ and positive integers $n_1|n_2|...|n_k$ define a group $(r, n_1, ..., n_k) = \mathbb{Z}^r \oplus \mathbb{Z}/n_1 \oplus ... \oplus \mathbb{Z}/n_k$. We proved that any finitely generated abelian group is isomorphic to a group of the form $(r, n_1, ..., n_k)$. Prove that if the groups $(r, n_1, ..., n_k)$ and $(s, m_1, ..., m_l)$ are isomorphic then $r = s$, $k = l$ and $n_i = m_i$ for all i. The numbers $n_1, ..., n_k$ are called the **invariant factors** and r is the **rank** (or torsion-free rank).

b) Find the rank and the invariant factors of the group $\mathbb{Z}^4/H$, where H is generated by $(-1, -2, -3, -4)$, $(3, 8, 5, 6)$, $(-1, 0, -13, -16)$, $(-3, -4, -13, -6)$.

Problem 2. a) Let F be a field. Show that the group $G = GL_n(F)$ has a composition series iff F is a finite field. Show the same for the a chief series (or principal series).

b) Find all composition series of the symmetric group S_n.

c) Show that a solvable group has a composition series iff it is finite.

Problem 3. a) Prove that if a group G has a composition series then it has a chief series. Is the converse true?

b) Suppose that G has a composition series. Show that every chief factor of G is a direct product of some finite number of copies of a simple group.