Homework 4

Solve problem 10 to section 5.2 and problems 9, 10, 11, 12, 13, 14, 15, 16 to section 5.4. Also solve the following problems:

Problem 1. a) Let G be a finite group. Prove that $\gamma_2 G \cap \zeta_1(G) \subseteq Frat(G)$.

b) Let G be a finite group and H a normal subgroup of G contained in $Frat(G)$. Prove that the group of inner automorphisms of H is contained in the Frattini subgroup of the group of all automorphisms of H.

c) Let $\mathcal{P}$ be a group-theoretical property of finite groups such that $G/Frat(G)$ has $\mathcal{P}$ iff G has $\mathcal{P}$. Prove that if G/H has $\mathcal{P}$ then G has a subgroup T such that T has $\mathcal{P}$ and $G = HT$. Conclude that if a finite group G has a normal subgroup H such that G/H is nilpotent, then G has a nilpotent subgroup T such that $G = HT$.

Problem 2. a) Let G be a group acting on a group H. Suppose that H has a normal series $H_1 = H \supseteq H_2 \supseteq ... \supseteq H_m = 1$ such that each H_i is $G-$stable (so this is a normal $G-$series). Suppose that G acts trivially on each factor H_i/H_{i+1}. Prove that $\gamma_j G$ acts trivially on H_i/H_{i+j} for all i and j (we set $H_t = 1$ for $t \geq m$). Conclude that if C is the centralizer of H in G (i.e. the set of all g which act trivially on H) then G/C is nilpotent of class at most $m-1$.

b) Suppose that $H_1 = H \supseteq H_2 \supseteq ... \supseteq H_m = 1$ is an arbitrary chain of subgroups of H (no assumptions about being normal or subnormal) and that G fixes each coset of H_{i+1} in H_i for all i. Show that G/C is nilpotent of class at most $(m-1)m/2$, where C is the centralizer of H in G.

Problem 3. Let A be a finite abelian normal subgroup of a group G such that $A \cap Frat(G) = 1$. Show that there is a subgroup $H < G$ such that $A \cap H = 1$ and $G = AH$ (we proved this in class under the assumption that G is finite).

Problem 4. Let G be a finite $p-$group of order p^n. Prove that G has an abelian normal subgroup of order p^m for some m such that $m(m+1) \geq 2n$.