Homework 5

Solve problems 3, 7, 8, 9 to section 11.1 and problems 2, 3, 8 to section 9.1. Also solve the following problems:

Problem 1. Let G be a group and H its subgroup of finite index n. Set H' for the derived subgroup of H. Choose a set $g_1, ..., g_n$ of left coset representatives of H in G. For every $g \in G$ there is unique permutation π of $\{1, 2, ..., n\}$ and unique elements h_i such t hat $gg_i = g_{\pi(i)}h_i$. Set $t(g) = h_1h_2...h_n$.

a) Show that the coset $t(g)H'$ of H' in H does not depend on the choice of the left coset representatives.

b) Show that the function $T_{G \mapsto H} = T : G \longrightarrow H/H'$ given by $T(g) = t(g)H'$ is a group homomorphism.

c) Show that the analogous construction with right cosets gives the same homomorphism.

The homomorphism T defined above is called **transfer** and it is an important tool in group theory. It has the following useful description. Recall that G acts on the left cosets of H in G by left multiplication. The action of $g \in G$ on these cosets of H splits the cosets into some number of orbits (cycles), say m orbits. Suppose that the $i-$th orbit consists of n_i cosets and let a_iH be one of them. Thus the $i-$th orbit consists of cosets $a_iH, ga_iH, ..., g^{n_i-1}a_iH$ and we have $g^{n_i}a_iH = a_iH$, i.e. $a_i^{-1}g^{n_i}a_i \in H$. Set $k_i = a_i^{-1}g^{n_i}a_i$.

d) Show that $T(g) = k_1k_2...k_mH'$.

e) Show that if K is a subgroup of finite index in H then $T_{H \mapsto K}T_{G \mapsto H} = T_{G \mapsto K}$.

f) Show that if g is in the center of G then $T(g) = g^nH'$ (note that $g^n \in H$).

The following problem illustrates the usefulness of the transfer.

Problem 2. Let G be a finite group and P its Sylow $p-$subgroup.

a) Suppose that a, b are in the center of P and that there is $g \in G$ such that $gag^{-1} = b$. Prove that there is $u \in G$ such that $uPu^{-1} = P$ (i.e. u normalizes

P) and $uau^{-1} = b$. Hint: Show first that both P and $g^{-1}Pg$ are contained in the centralizer of a.

b) Suppose further that P is in the center of its normalizer N in G (in particular, P is abelian). Show that if $p \in P$ and $gpg^{-1} \in P$ for some $g \in G$ then $gpg^{-1} = p$.

c) Under the assumptions of b) show that the transfer T from G to P maps any $p \in P$ to p^n, where n is the index of P in G. Conclude that T is surjective.

d) Deduce from c) that G has a normal subgroup H of order n such that $G = PH$.

e) Show that if p is the smallest prime divisor of the order of G and P is cyclic then the assumptions of b) are satisfied, so G has a normal subgroup H such that $H \cap P = 1$ and $G = HP$. In particular, G is not simple.

f) Show that if all Sylow subgroups of a finite group G are cyclic then G is solvable.

Remark. In fact much more can be proved (and it is not that hard): G has two elements a, b of orders m,n respectively such that $b^{-1}ab = a^r$ for some r such that $m|(r^n - 1)$ and $(m, n(r-1)) = 1$. Moreover G has order mn, $G' = < a >$ and G/G' is cyclic of order n. It is easy to see that every m, n, r as above correspond to a unique group with cyclic Sylow subgroups. In particular, this gives a full classification of groups of order N for any square-free N.

Problem 3. Let H be a $p-$group which acts on an abelian $p-$group N. Suppose that $\mathrm{H}^1(H, N) = 0$. Prove that $\mathrm{H}^1(K, N) = 0 = \mathrm{H}^2(K, N)$ for every subgroup K of H