Homework 1

Problem 1. Let $\mathcal{C}$, $\mathcal{D}$ be categories. Two functors $F : \mathcal{C} \longrightarrow \mathcal{D}$, $G : \mathcal{D} \longrightarrow \mathcal{C}$ are called **adjoint** if the following functors $T_F, T_G : \mathcal{C}^o \times \mathcal{D} \longrightarrow SET$ are naturally equivalent:

$T_F(X,Y) = Mor_{\mathcal{D}}(F(X),Y)$

$T_G(X,Y) = Mor_{\mathcal{C}}(X,G(Y))$

(it is clear how these functors are defined on morphisms). In other words, for every $X \in Ob\mathcal{C}$ and $Y \in Ob\mathcal{D}$ there is a bijection $f_{X,Y} : Mor_{\mathcal{D}}(F(X),Y) \longrightarrow Mor_{\mathcal{C}}(X,G(Y))$ and these bijections are compatible with composition of morphisms. We say that F is **left adjoint** to G and G is **right adjoint** to F.

a) Show that if a right adjoint to F exists, then it is unique up to natural equivalence; same for left adjoint to G.

b) Let G be the forgetful functor from the category of abelian groups to the category of sets. Show that it has a left adjoint functor. Do the same for the forgetful functors from abelian groups to groups and from topological spaces to sets.

c) Prove that the forgetful functor from abelian groups to groups does not have a right adjoint.

d) Prove that if F, G are adjoint functors (F is left adjoint to G) then F preserves colimits and G preserves limits (so, for example, if $T : \mathcal{S} \longrightarrow \mathcal{C}$ is a functor from a small category and $colimT$ exists, then $colimFT = F(colimT)$). This and b) should explain why the limits in the categories of sets, groups and topological spaces look the same. Conclude that if $\mathcal{C}$, $\mathcal{D}$ are abelian categories then F is right exact and G is left exact.

e) Let $\mathcal{S}$ be a small category and $\mathcal{C}$ be a category such that $colimF$ exists for every functor $F : \mathcal{S} \longrightarrow \mathcal{C}$. One can define a functor $colim$ from the category of functors $Fun(\mathcal{S},\mathcal{C})$ to $\mathcal{C}$ by sending F to $colimF$ (some choices need to be made here). There is also a functor h from $\mathcal{C}$ to $Fun(\mathcal{S},\mathcal{C})$ which takes an object X to the constant functor h_X. Show that $colim$ is left adjoint to h. Formulate and prove similar result for lim.

Problem 2. Recall that a functor $F : \mathcal{C} \longrightarrow SET$ is called representable if it is naturally equivalent to a functor $R_X = Mor(X, -)$ for some object X (so $R_X(Y) = Mor(X, Y)$ and $R_X(f)$ maps $g \in Mor(X, Y)$ to $fg \in Mor(X, Z)$ for any $f \in Mor(Y, Z)$). Let $G : \mathcal{C} \longrightarrow SET$ be any functor (not necessarily representable). If (η_Y) is a natural transformation from R_X to G (so $\eta_Y : R_X(Y) \longrightarrow G(Y)$), we can assign to it an element $\eta_X(id_X)$ of $G(X)$. Show that this defines a bijection between natural transformations from R_X to G and $G(X)$. Conclude that R_X and R_Y are naturally equivalent iff X and Y are isomorphic. Formulate and prove similar result for $Mor(-, X)$. This result is called **Yoneda's Lemma**.

Problem 3. a) Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between additive categories. Prove that the following conditions are equivalent:

1. F is additive
2. F preserves products of any two objects

b) Suppose that in a) the categories are abelian. Prove that if F is left exact or right exact then it is additive.

Problem 4. a) For a morphism $f : X \longrightarrow Y$ define $d(f) = X$ (domain of f) and $r(f) = Y$. Let $F : \mathcal{S} \longrightarrow \mathcal{C}$ be a functor from a small category $\mathcal{S}$. Suppose that the product $A = \prod_X F(X)$ exists (product over all objects of $\mathcal{S}$) and also that the product $B = \prod_f F(r(f))$ exists (product over all morphisms of $\mathcal{S}$). For every X we have the projection $p_X : A \longrightarrow F(X)$. For every f we define two morphisms i_f, j_f from A to $F(r(f))$ by $i_f = p_{r(f)}$ and $j_f = F(f)p_{d(f)}$. By the definition of product, there exist morphisms i, j from A to B such that $\pi_f i = i_f$ and $\pi_f j = j_f$, where π_f is the projection of B to $F(r(f))$. Prove that $limF$ is the same as the equalizer of i, j. Conclude that if equalizers and products of two objects (all objects) exists in $\mathcal{C}$ then all finite limits (resp. all limits) exist in $\mathcal{C}$. Conclude that all finite limits exist in an abelian category (a limit is called finite if the objects of $\mathcal{S}$ form a finite set). Formulate and prove similar result for colimits.

b) Use a) to conlude that a left (right) exact functor between abelian categories preserves all finite limits (colimits).

Remark. Let $\mathcal{S}$ be a category with two objects a, b and two morphisms u, v from a to b (plus the identity morphisms). If $\mathcal{C}$ is a category and we have two morphisms $f, g : X \longrightarrow Y$ in $\mathcal{C}$, then we may define a funcor $F : \mathcal{S} \longrightarrow \mathcal{C}$ by $F(a) = X$, $F(b) = Y$, $F(u) = f$, $F(v) = g$. The equalizer of f, g is by definition the limit $limF$, and the coequalizer is the colimit $colimF$.