Homework 3

Problem 1. a) Let G, H be groups, let P, M be RG modules and let Q, N be RH modules. Recall that in our discussion of cross product we defined a homomorphism (called cross product)

$$\mathrm{Hom}_{RG}(P,M)\otimes_R \mathrm{Hom}_{RH}(Q,M) \longrightarrow \mathrm{Hom}_{R(G\times H)}(P\otimes_R Q, M\otimes_R N)$$

which takes $f\otimes g$ (where $f\in \mathrm{Hom}_{RG}(P,M)$ and $g\in \mathrm{Hom}_{RH}(Q,M)$) to $f\times g$ defined by $(f\times g)(p\otimes q)=f(p)\otimes g(q)$. Prove that if P, Q are finitely generated projective modules then the cross product is an isomorphism. (Hint: Prove it first for free modules).

b) Suppose now that G, H are groups such that R has resolutions $P_\bullet$, $Q_\bullet$ such that P_i (resp. Q_i) are finitely generated projective RG (resp RH) modules (this holds for finite groups, finitely generated abelian groups, etc.). Using these resolutions, a) and Künneth's theorem prove that if R is a PID (or, more generally, a hereditary ring) then there are functorial exact sequences

$$0\longrightarrow \bigoplus_{i+j=n} H^i(G,M)\otimes_R H^j(H,N)\longrightarrow H^n(G\times H, M\otimes_R N)\longrightarrow$$

$$\longrightarrow \bigoplus_{i+j=n+1} \mathrm{Tor}_1^R(H^i(G,M),H^j(H,N))\longrightarrow 0$$

c) In the situation when all the Tor_1 in b) vanish the cross product gives an isomorphism

$$H^*(G,M)\otimes_R H^*(H,N)\approx H^*(G\times H, M\otimes_R N)$$

Show that this is an isomorphism of graded algebras. Use this to compute the ring $H^*(G,\mathbb{Z})$ for a free abelian group of finite rank G.

Read carefully Chapter V of Brown's book (you should also be familiar with the content of Chapter IV). Solve Problems 1, 4, 5 to V.3, Problem 3 to III.1 and Problem 1 to V.2.