Homework 5

due on Wednesday, November 12

Read Chapter 12 of Dummit and Foote.

Problem 1. Let R be a PID. Suppose that M is a torsion R-module and $m \in M$ is such that $\mathrm{ann}(m) = (r) = \mathrm{ann}(M)$.

a) Show that in the set of all submodules of M which intersect $< m >$ trivially there is a maximal element N (with respect to inclusion). The next steps will show that M is the direct sum of $< m >$ and N. Show that this is equivalent to the statement that $M/N =< m + N >$ is a cyclic module.

b) Show that $\mathrm{ann}(m + N) = (r) = \mathrm{ann}(M/N)$ and for every non-zero element $x \in M/N$ the cyclic modules $< x >$ and $< m + N >$ have non-trivial intersection.

c) For $0 \neq n \in M$ consider the set $I = \{a \in R : an \in < m >\}$. Show that it is an ideal of R which contains r. Let b be a generator of I, so $b|r$. Note that $bn = cm$ for some $c \in R$. Show that r divides $(r/b)c$ and conclude that $b|c$. Prove that $< m > \cap < n - (c/b)m >= \{0\}$.

d) Use c) to show that in b) we have $M/N =< m + N >$.

e) Show that if M is finitely generated then m with the required property exists (do not use the results about decomposition into a direct sum of cyclic modules).

f) Suppose that $m, m' \in M$ both satisfy the assumptions of the problem. prove that there is an automorphism ϕ of M such that $\phi(m) = m'$.

Problem 1 gives a different proof of the fact that a finitely generated torsion module is a direct sum of cyclic modules.

Problem 2. Find the rank, the invariant factors and the elementary divisors of the group $\mathbb{Z}^4/H$, where H is generated by $(-1, -2, -3, -4)$, $(3, 8, 5, 6)$, $(-1, 0, -13, -16)$, $(-3, -4, -13, -6)$. (Find a compatible bases of $\mathbb{Z}^4$ and H).

Problem 3. Let A be a square matrix with entries in a field K. Prove that A and A^t are similar, where A^t is the transpose of A.

Problem 4. Let A be an $m \times n$ matrix with entries in a PID R. We proved that there are invertible matrices Y, Z such that YAZ is a diagonal matrix with diagonal entries $a_1, ..., a_s, 0, 0, ...$, where $a_1|a_2|...|a_s \neq 0$. Furthermore, $a_1, ..., a_s$ are uniquely determined up to invertible elements. Reacall that a minor of size k of a matrix is the determinant of some $k \times k$ square submatrix. Prove that for each k, the product $a_1...a_k$ is a greatest common divisor of all $k \times k$ minors of A. (This provides a way to compute the invariant factors).