Homework 6

due on Friday, December 12

Problem 1. Let R be a ring. Suppose that I is a left ideal of R minimal among non-zero left ideals. Prove that either $I^2 = 0$ or there is idempotent $e \in I$ such that $I = Re$.

Problem 2. Let R be a ring and let $J = J(R)$ be the Jacobson radical of R.

a) Prove that $J(R[[x]]) = JR[[x]] + (x)R[[x]]$.

b) Prove that $J(M_n(R)) = M_n(J)$. Conclude that if I is an ideal contained in J then $A \in M_n(R)$ is invertible iff its image in $M_n(R/I)$ is invertible.

c) Prove that if R is commutative then $J(R[x]) = N[x]$, where N is the nilradical of R.

d) Prove that if R has no non-zero nil ideals (i.e. two sided ideals whose all elements are nilpotent) then $J(R[x]) = \{0\}$.

e) Prove that $JM \subseteq \mathrm{rad}(M)$ for any left R-module M. Prove that the equality holds for projective modules M. Hint: Show that $\mathrm{rad}(M \oplus N) = \mathrm{rad}(M) \oplus \mathrm{rad}(N)$.

Problem 3. Let M be a left R-module and f and endomorphism of M. For each n let K_n, I_n be the kernel and image of f^n respectively.

a) Prove that if $I_n = I_{n+1}$ then $M = I_n + K_n$.

b) Prove that if $K_n = K_{n+1}$ then $I_n \cap K_n = \{0\}$.

c) Prove that if M is Artinian then $M = I_n + K_n$ for all sufficiently large n.

d) Prove that if M is Noetherian then $I_n \cap K_n = \{0\}$ for all sufficiently large n.

e) Prove that if M is Artinian and f is a monomorphism then f is an isomorphism.

f) Prove that if M is Noetherian and f is surjective then f is an isomorphism. (Remark: If R is commutative this is true for any finitely generated module M. Can you prove it?)

g) Suppose that M has finite length and cannot be decomposed into a direct sum of proper submodules. Prove that either f is nilpotent or it is an isomorphism. Conclude that in the ring $\mathrm{End}_R(M)$ the Jacobson radical J consists of nilpotent elements and $\mathrm{End}_R(M)/J$ is a division ring.

Problem 4. Let R be a ring and let I be a two-sided nil-ideal of R (i.e. every element of I is nilpotent).

a) Let $a \in R$ and let n be a positive integer. Show that $(1-a)^n = 1 - ab_n$ for some $b_n \in R$ which commutes with a.

b) Suppose that $(a-a^2)^n = 0$. Show that $a^n = a^n(ab_n)$ and conclude that $e = a^n b_n^n$ is an idempotent. Suppose that $a(1-a) \in K$ for some two sided ideal K of R. Show that $e - a \in K$.

c) Suppose that $a + I$ is an idempotent of R/I. Prove that there is an idempotent e of R such that $e + I = a + I$. We say that idemptents of R/I can be lifted to idempotents of R.

d) Suppose that $a_i + I$, $i = 1, 2, ..., k$ are idempotents of R/I such that $a_i a_j \in I$ for $i \neq j$ (i.e. the idempotents are orthogonal). Prove that there are idempotents $e_i \in R$ such that $e_i e_j = 0$ for $i \neq j$ and $a_i + I = e_i + I$.

e) Suppose that R is left Artinian. Show that any non-nilpotent left ideal K of R contains a non-zero idempotent. Hint: go modulo the Jacobson radical. Note that in b) if a belongs to some left ideal then so does e.

f)* Show that if R is left Artinian then any left ideal is of the form $Re \oplus T$ where e is an idempotent and T is a nilpotent left ideal.

Problem 5. Let $f : P \longrightarrow Q$ be a homomorphism of finitely generated projective (left) R-modules. Suppose that the induced map $P/JP \longrightarrow Q/JQ$ is an isomorphism (here J is the Jacobson radical). Prove that f is an isomorphism. Show that any homomorphism $P/JP \longrightarrow Q/JQ$ lifts to a homomorphism $P \longrightarrow Q$. Conclude that if P is finitely generated projective and P/JP is a free R/J-module then P is a free R-module. Conclude that if R/J is a division ring (such rings are called **local**) then every finitely generated projective R-module is free. Hint: Use Nakayama's lemma.