Homework 1
due on Friday, September 9

Problem 1. An element a of a ring R is called **nilpotent** if $a^m = 0$ for some $m > 0$.

a) Prove that in a commutative ring R the set N of all nilpotent elements of R is an ideal. This ideal is called the **nilradical** of R. Prove that 0 is the only nilpotent element of R/N.

b) Let R be a commutative ring and let $a_1, ..., a_n \in R$ be nilpotent. Set I for the ideal $< a_1, ..., a_n >$ generated by $a_1, ..., a_n$. Prove that there is a positive integer N such that $x_1 x_2 ... x_N = 0$ for any $x_1, ..., x_N$ in I (i.e. that $I^N = 0$).

c) Prove that the set of all nilpotent elements in the ring $M_2(\mathbb{R})$ is not an ideal.

d) Prove that if p is a prime and $m > 0$ then every element of $\mathbb{Z}/p^m\mathbb{Z}$ is either nilpotent or invertible.

e) Find the nilradical of $\mathbb{Z}/36\mathbb{Z}$ (by correspondence theorem, it is equal to $n\mathbb{Z}/36\mathbb{Z}$ for some n).

Problem 2. Let R be a commutative ring. For an ideal I of R define

$$\sqrt{I} = \{x \in R : x^n \in I \text{ for some } n > 0\}.$$

a) Prove that $\sqrt{I}$ is an ideal. It is called the **radical** of I.

b) Prove that $\sqrt{\{0\}}$ is the nilradical of R.

c) Consider a surjective homomorphism $f : R \longrightarrow S$. Prove that in the correspondence theorem the nilradical of S corresponds to $\sqrt{\ker f}$.

d) Prove that $R/\sqrt{I}$ has trivial nilradical.

Problem 3. A subset S of a commutative ring is called **multiplicative** if $0 \notin S$ and for any $a, b \in S$ also $ab \in S$. .

a) Let I be an ideal of a commutatuve unital ring R. Prove that I is a prime ideal iff $R - I$ is multiplicative.

b) Let S be a multiplicative subset of a comutative unital ring R. Consider the set T of all ideals of R which are disjoint with S. Prove that this set contains maximal elements (with respect to inclusion; this requires Zorn's Lemma and is very similar to the proof that every ring has a maximal ideal). Prove that every maximal element of T is a prime ideal.

c) Use b) to prove that if $a \in R$ is not nilpotent then there is a prime ideal in R which does not contain a.

d) Prove that the nilradical of a commutative unital ring R coincides with the intersection of all prime ideals.

Problem 4. Let $f : R \longrightarrow S$ be a homomorphism of commutative unital rings.

a) Prove that if P is a prime ideal of S then $f^{-1}(P)$ is a prime ideal of R. Is this true for non-commutative rings?

b) Find an example when P is a maximal ideal of S but $f^{-1}(P)$ is not maximal in R.

c) Prove that if f is onto and Q is a prime ideal of R such that $\ker f \subseteq Q$ then $f(Q)$ is a prime ideal of S. Is this true for non-commutative rings?

d) Suppose that f is surjective. Prove that if P is a maximal ideal of S then $f^{-1}(P)$ is maximal in R. Prove that if Q is a maximal ideal of R then $f(Q)$ is either S or it is a maximal ideal of S. Show by example that a similar statement for prime ideals is false.

e) Find all prime ideals of $\mathbb{Z}/36\mathbb{Z}$.

Problem 5. Let F be a finite field (for example, $F = \mathbb{Z}/p\mathbb{Z}$, where p is a prime). Consider the subset R of the ring $\mathrm{M}_3(F)$ of all 3×3 matrices over F which consists of all matrices of the form $\begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & d & e \end{pmatrix}$. Let N be the subsets of R of all matrices with $a = 0 = e$ and let M be the subset of N of all matrices such that $a = b = d = e = 0$.

a) Prove that R, N, M are subrings of $\mathrm{M}_3(F)$ (in the category of non-unital rings).

b) Prove that $N \lhd R$, $M = N^2$, and $MN = NM = \{0\}$ (i.e. $N^3 = \{0\}$).

c) Prove that R/N is isomorphic to the ring $F \oplus F$ (with componentwise addition and multiplication).

d) Prove that $R^2 = R$ but $RN \neq N$ and $NR \neq N$. Conclude that R has no left or right identity.

e) Prove that if $u \in R$ is non-zero then uR and Ru are non-zero.

f) Prove that there is an element $w \in R/M$ such that $w(R/M) = \{0\}$ and an element $v \in R/M$ such that $(R/M)v = \{0\}$.

Remark. We took F finite so that R is a finite ring. But the problem remains true for any field F.