Homework 5
due on Wednesday, December 7

Read Chapter 12 of Dummit and Foote.

Problem 1. Let R be a commutative ring and S a multiplicative subset of R. Prove that $S^{-1}R$ is a flat R-module. Show that for any R-module M we have $S^{-1}M \simeq S^{-1}R \otimes_R M$.

Problem 2. Let R be a commutatime ring with unique maximal ideal J (such rings are called **local**).

a) Prove that if K is a submodule of R^n such that $K + JR^n = R^n$ then $K = R^n$.

b) Prove that if M is a finitely generated R-module such that $JM = M$ then $M = 0$. Hint: consider a presentation of M. This result is called **Nakayama Lemma**.

c) Prove that if $f : M \longrightarrow N$ is a homomorphism of finitely generated R-modules which iduces a surjective map $M/JM \longrightarrow N/JN$ then f is surjective. Hint: Tensoring with R/J is right exact.

d) Let $f : P \longrightarrow Q$ be a homomorphism of finitely generated projective R-modules such that the induced map $P/JP \longrightarrow Q/JQ$ is an isomorphism. Prove that f is an isomorphism.

e) Let P be a finitely generated projective R-module. Let $a_1, ..., a_k$ be elements of P such that $a_1 + JP, ..., a_k + JP$ is a basis of the R/J-vector space P/JP. Prove that $a_1, ..., a_k$ is a basis of P. Conclude that every finitely generated projective R-module is free.

Problem 3. Let R be a PID. Prove that $(a_1, ..., a_n) \in R^n$ is a row of an invertible $n \times n$ matrix iff $\gcd(a_1, ..., a_n) = 1$.

Problem 4. Let R be a PID. Let M be a submodule of R^n such that R^n/M is torsion. Prove that there is a basis $m_1, ..., m_n$ of M of the form $m_1 = (a_{11}, 0, ..., 0)$, $m_2 = (a_{21}, a_{22}, 0, ..., 0)$, $m_3 = (a_{31}, a_{32}, a_{33}, 0, ..., 0)$,... , $m_n = (a_{n1}, ..., a_{nn})$ (so the matrix with columns $m_1, ..., m_n$ is upper triangular). Prove that when $R = \mathbb{Z}$ then there is unique such basis which satisfies $0 \le a_{i,j} < a_{ii}$ for every i and every $j > i$.

Problem 5. Let R be a PID. Suppose that M is a torsion R-module and $m \in M$ is such that $\mathrm{ann}(m) = (r) = \mathrm{ann}(M)$.

a) Show that in the set of all submodules of M which intersect $< m >$ trivially there is a maximal element N (with respect to inclusion). The next steps will show that M is the direct sum of $< m >$ and N. Show that this is equivalent to the statement that $M/N =< m + N >$ is a cyclic module.

b) Show that $\mathrm{ann}(m + N) = (r) = \mathrm{ann}(M/N)$ and for every non-zero element $x \in M/N$ the cyclic modules $< x >$ and $< m + N >$ have non-trivial intersection.

c) For $0 \neq n \in M$ consider the set $I = \{a \in R : an \in < m >\}$. Show that it is an ideal of R which contains r. Let b be a generator of I, so $b|r$. Note that $bn = cm$ for some $c \in R$. Show that r divides $(r/b)c$ and conclude that $b|c$. Prove that $< m > \cap < n - (c/b)m >= \{0\}$.

d) Use c) to show that in b) we have $M/N =< m + N >$.

e) Show that if M is finitely generated then m with the required property exists (do not use the results about decomposition into a direct sum of cyclic modules).

f) Suppose that $m, m' \in M$ both satisfy the assumptions of the problem. prove that there is an automorphism ϕ of M such that $\phi(m) = m'$.

Problem 1 gives a different proof of the fact that a finitely generated torsion module is a direct sum of cyclic modules.

Problem 6. Find the rank, the invariant factors and the elementary divisors of the group $\mathbb{Z}^4/H$, where H is generated by $(-1, -2, -3, -4)$, $(3, 8, 5, 6)$, $(-1, 0, -13, -16)$, $(-3, -4, -13, -6)$. (Find a compatible bases of $\mathbb{Z}^4$ and H).

Problem 7. Let A be a square matrix with entries in a field K. Prove that A and A^t are similar, where A^t is the transpose of A.

Problem 8. Let A be an $m \times n$ matrix with entries in a PID R. We proved that there are invertible matrices Y, Z such that YAZ is a diagonal matrix with diagonal entries $a_1, ..., a_s, 0, 0, ...,$ where $a_1|a_2|...|a_s \neq 0$. Furthermore, $a_1, ..., a_s$ are uniquely determined up to invertible elements. Reacall that a minor of size k of a matrix is the determinant of some $k \times k$ square submatrix. Prove that for each k, the product $a_1 ... a_k$ is a greatest common divisor of all $k \times k$ minors of A. (This provides a way to compute the invariant factors).