Midterm, Math 525

November 4

Solve 4 of the following problems (each is worth 10 points). You may solve more for extra credit. State any main result you are using in your solution.

Problem 1. Let $R = \mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} : a, b \in \mathbb{Z}\}$ (so this ring is a subring of the Eisenstein integers).

a) Prove that $1, -1$ are the only invertible elements in R. (2 points)

b) Prove that 2, $1 + \sqrt{-3}$, $1 - \sqrt{-3}$ are irreducible in R. Conclude that R is not UFD (find 2 inequivalent factorizations of 4). (4 points)

c) Prove that the ideal $I =< 2, 1 + \sqrt{-3} >$ of R is not principal and that it is maximal. (4 points)

Problem 2. Prove that if R is a commutative integral domain which does not have infinite strictly ascending chains of principal ideals then the same is true about $R[x]$.

Problem 3. Let R be a ring and let $a \in R$. Prove that the following conditions are equivalent:

1. a belongs to every maximal left ideal of R.
2. $1 + ra$ has a left inverse for every $r \in R$.
3. $1 + ra$ is invertible for every $r \in R$.

Problem 4. Let I be an ideal in a commutative ring R and let P_1, P_2, P_2 be prime ideals of R such that $I \subseteq P_1 \cup P_2 \cup P_3$. Prove that $I \subseteq P_i$ for some i.
Hint. Prove this first for 2 prime ideals. Can you generalize it to n prime ideals?

Problem 5. Prove that the polynomial $f = xy^{2011} + x^{2011}y + x - y - 1$ is irreducible in $\mathbb{Q}[x, y]$. Hint: Consider f as a polynomial in $K[y]$, where $K = \mathbb{Q}(x)$ is the field of fractions of $\mathbb{Q}[x]$.

Problem 6. Let R be a PID and let I, J be proper ideals of R.

a) Prove that the intersection of all the ideals I^n, $n = 1, 2, \ldots$, is trivial (this is true, but harder to prove, for any commutative Noetherian integral domain and any ideal I). (4 points)

b) Prove that if J is not trivial then $\bigcap_{n=1}^{\infty}(J + I^n) = J + I^k$ for some k. (6 points)