Homework 5
due on Wednesday, December 11

Read Chapter 12 of Dummit and Foote.

Problem 1. Let R be a commutative ring and S a multiplicative subset of R. Show that for any R-module M we have $S^{-1}M \simeq S^{-1}R \otimes_R M$. Prove that $S^{-1}R$ is a flat R-module.

Problem 2. Let R be a ring. Let J be the intersection of all maximal left ideals of R (so J is a left ideal of R).

a) Prove that if $a \in J$ then $1 + a$ has a left inverse u and $1 - u \in J$. Conclude that $1 + a$ is invertible.

b) Prove that if $r \in R$ then the left ideal Jr is contained in every maximal left ideal of R. Conclude that J is a two-sided ideal.

c) Prove that J is contained in every maximal right ideal of R. Conclude that J is the inersection of all maximal right ideals of R.

d) The ideal J is called the **Jacobson radical** of R. Let $a \in R$. Prove that the following conditions are equivalent:

1. $a \in J$.
2. $1 - ra$ has a left inverse for all $r \in R$.
3. $1 - ar$ has a right inverse for all $r \in R$.
4. $1 - ras$ is invertible for every $r, s \in R$.
5. $aM = 0$ for every simple left R-module M.
6. $Ma = 0$ for every simple right R-module M.

Hint: Show that a simple left R-module is isomorphic to R/K for some maximal left ideal K.

Problem 3. Let R be a ring and J its Jacobson radical.

a) Suppose that M is a finitely generated left R-module such that $JM = M$. Prove that $M = 0$ Hint: work with a minimal set of generators of M. This result is called **Nakayama's Lemma**.

b) Suppose that N is a submodule of a finitely generated left R-module M such that $M = N + JM$. Prove that $M = N$.

c) Let $f : M \longrightarrow N$ be a homomorphism of feft R-modules. It induces a homomorphism $\overline{f} : M/JM \longrightarrow N/JN$ of R/J-modules. Prove that if N is finitely generated then f is surjective iff $\overline{f}$ is surjective. Prove that if in addition M is finitely generated and f is a split epimorphism then f is an isomorphism iff $\overline{f}$ is an isomorphism.

d) Suppose that P, Q are left R-modules and $g : P/JP \longrightarrow Q/JQ$ is a surjective homomorphism of R/J-modules. Prove that if P is projective then there is a homomorphism $f : P \longrightarrow Q$ such that $\overline{f} = g$. Conclude that if both P, Q are finitely generarated projective left R-modules then P and Q are isomorphic if and only if P/JP and Q/JQ are isomorphic.

e) Let R be a ring such that R/J is a division ring (such rings are called **local**; this means that R has unique maximal left ideal). Prove that every finitely generated projective R-module is free.

Remark: Kaplansky proved that the same holds for all projective modules (not necessarily finitely generated).

Problem 4. Let $0 \longrightarrow F_n \longrightarrow \ldots \longrightarrow F_0 \longrightarrow P \longrightarrow 0$ be an exact sequence of finitely generated R-modules, where P is projective and F_i are free. Prove that $P \oplus R^n$ is free for some n (such P are called **stably free**).

Problem 5. Let R be a UFD. Prove that an ideal of R is projective iff it is principal.

Problem 6. Let R be a commutative ring such that every finitely generated projective R-module is free. Prove that $(a_1, ..., a_n) \in R^n$ is a row of an invertible $n \times n$ matrix over R iff $(a_1, ..., a_n) = R$. Hint: Note that the condition is equivalent to $(a_1, ..., a_n)$ being part of a basis of R^n. Pick $x_1, ..., x_n$ such that $x_1 a_1 + \ldots x_n a_n = 1$ and consider the homomorphism $R^n \longrightarrow R$ sending $(b_1, \ldots, b_n)$ to $x_1 b_1 + \ldots x_n b_n$.

Problem 7. Let R be a PID and I a non-zero ideal of R. Prove that R/I is and injective R/I-module.

Problem 8. Let R be a ring with a striclty increasing chain of right ideals $J_1 \subsetneq J_2 \subsetneq \ldots$. Let $J = \bigcup_{i=0}^{\infty} J_i$. Let M_i be an injective right R-module containing J/J_i for $i = 1, 2, \ldots$. Prove that the module $M = \bigoplus_{i=1}^{\infty} M_i$ is not injective. Conclude that if a direct sum of any countable set of injective right R-modules is injective then R is right Noetherian.

Hinf: Define a homomorphism from J to M which can not be lifted to R. Note that any homomorhism from R to M has image contained in a direct sum of finitely many of the M_i's.

Problem 9. Let $R = \mathbb{Z}[x]$. Let $M = \mathbb{Q}(x)/\mathbb{Z}[x]$, where $\mathbb{Q}(x)$ is the field of rational functions with rational coefficients. Prove that M is a divisible $\mathbb{Z}[x]$-module. Consider the ideal $I = (2, x)$ of $\mathbb{Z}[x]$. Prove that there is a $\mathbb{Z}[x]$-module homomorphism $f : I \longrightarrow M$ such that $f(2) = \frac{1}{x} + \mathbb{Z}[x]$ and $f(x) = \frac{x}{2} + \mathbb{Z}[x]$. Use it to prove that M is not injective $\mathbb{Z}[x]$-module.

Problem 10. Let R be a PID. Let M be a submodule of R^n such that R^n/M is torsion. Prove that there is a basis $m_1, ..., m_n$ of M of the form $m_1 = (a_{11}, 0, ..., 0)$, $m_2 = (a_{21}, a_{22}, 0, ..., 0)$, $m_3 = (a_{31}, a_{32}, a_{33}, 0, ..., 0)$,... , $m_n = (a_{n1}, ..., a_{nn})$ (so the matrix with columns $m_1, ..., m_n$ is upper triangular). Prove that when $R = \mathbb{Z}$ then there is unique such basis which satisfies $0 \leq a_{i,j} < a_{ii}$ for every i and every $j > i$.

Problem 11. Let R be a PID. Suppose that M is a torsion R-module and $m \in M$ is such that $\text{ann}(m) = (r) = \text{ann}(M)$.

a) Show that in the set of all submodules of M which intersect $< m >$ trivially there is a maximal element N (with respect to inclusion). The next steps will show that M is the direct sum of $< m >$ and N. Show that this is equivalent to the statement that $M/N =< m + N >$ is a cyclic module.

b) Show that $\text{ann}(m + N) = (r) = \text{ann}(M/N)$ and for every non-zero element $x \in M/N$ the cyclic modules $< x >$ and $< m + N >$ have non-trivial intersection.

c) For $0 \neq n \in M$ consider the set $I = \{a \in R : an \in< m >\}$. Show that it is an ideal of R which contains r. Let b be a generator of I, so $b|r$. Note that $bn = cm$ for some $c \in R$. Show that r divides $(r/b)c$ and conclude that $b|c$. Prove that

$< m > \cap < n - (c/b)m >= \{0\}$.

d) Use c) to show that in b) we have $M/N =< m + N >$.

e) Show that if M is finitely generated then m with the required property exists (do not use the results about decomposition into a direct sum of cyclic modules).

f) Suppose that $m, m' \in M$ both satisfy the assumptions of the problem. prove that there is an automorphism ϕ of M such that $\phi(m) = m'$.

Problem 11 gives a different proof of the fact that a finitely generated torsion module is a direct sum of cyclic modules.