Homework 2

due on Monday, September 18

Study Chapters 7 and 8 of Dummit and Foote. Solve problem 26 to 7.1, problem 39 to 7.4 and the following problems.

Problem 1. Let R be a unique factorization domain. Let a, b, c be non-zero elements of R. Prove the following:

1. If $c|ab$ and $\gcd(a, c) = 1$ then $c|b$.
2. If $a|c$, $b|c$, and $\gcd(a, b) = 1$ then $ab|c$.
3. If $\gcd(a, c) = 1 = \gcd(b, c)$ then $\gcd(ab, c) = 1$.
4. If $c|a$ and $c|b$ then $c \gcd(a/c, b/c) = \gcd(a, b)$.
5. If m, n are positive integers then $\gcd(a, b) = 1$ iff $\gcd(a^m, b^n) = 1$.
6. If n is a positive integer and $a^n|b^n$ then $a|b$.
7. $\gcd(a, b)\mathrm{lcm}(a, b)$ is associated to ab.
8. $\gcd(a, b, c) = \gcd(a, \gcd(b, c))$ and $\mathrm{lcm}(a, b, c) = \mathrm{lcm}(a, \mathrm{lcm}(b, c))$.

Problem 2. Let I be an ideal of the ring R. Define $I[x]$ as the subset of $R[x]$ which consists of all the polynomials in $R[x]$ whose all coefficients belong to I. Prove that $I[x]$ is an ideal of $R[x]$ and that $R[x]/I[x]$ is naturally isomorphic to the polynomial ring $(R/I)[x]$.

Problem 3. Let R be a commutative ring and let $R[x]$ be the ring of polynomials in x with coefficients in R. Let $f = f_0 + f_1 x + ... + f_n x^n \in R[x]$. Prove that

a) f is invertible iff $f_0 \in R^\times$ and $f_1, ..., f_n$ are nilpotent.

b) f is nilpotent iff $f_0, ..., f_n$ are nilpotent.

c) f is a zero divisor iff $af = 0$ for some $0 \neq a \in R$.

d) Let P be a prime ideal of R and $f, g \in R[x]$. Prove that all coefficients of fg belong to P iff either all coefficients of f or all coefficients of g belong to P.

e) If f belongs to every maxiaml ideal of $R[x]$ then f is nilpotent.

Problem 4. Ler R be an integral domain.

a) Let $f, g \in R[x]$ be such that $fg = cx^n$ for some n and some $c \in R$, $c \neq 0$. Prove that there exist elements $a, b \in R$ and $m \leq n$ such that $f = ax^m$ and $g = bx^{n-m}$ and $ab = c$.

b) Suppose that $f = f_0 + f_1 x + ... + f_n x^n \in R[x]$. Suppose that there is a prime ideal P of R such that $f_n \notin P$, $f_0, ..., f_{n-1} \in P$ and $f_0 \notin P^2$. Prove that if $f = gh$ for some $g, h \in R[x]$ then one of g, h is constant. Conclude that if in addition f is monic then it is irreducible in $R[x]$. This result is known as **Eisenstein criterion**. Hint: Assume that $f = gh$ and both g, h have positive degree. Pass to the ring $(R/P)[x]$ and apply a) to show that constant terms of g and h belong to P. Derive contradiction.

c) Prove that the polynomial $2x^{10} + 21x^8 - 35x^2 + 14$ is irreducible in $\mathbb{Z}[x]$. Hint: Apply Eisenstein criterion with appropriate prime ideal P.

Problem 5. Find a greatest common divisor $d(x)$ of the polynomials $p(x) = x^3 + 4x^2 + x - 6$ and $q(x) = x^5 - 6x + 5$ in the ring $\mathbb{Q}[x]$ and find $a(x), b(x) \in \mathbb{Q}[x]$ such that $d(x) = a(x)p(x) + b(x)q(x)$.

Problem 6. Let $K \subseteq L$ be fields. Suppose that $f, g \in K[x]$ and $f|g$ in the ring $L[x]$. Prove that $f|g$ in the ring $K[x]$.