Homework 6

due on Wednesday, November 15

Read Chapter 10 of Dummit and Foote. Solve problems 14, 27, 28 to 10.5. Also solve the following problems.

Problem 1. Let R be a ring. Let J be the intersection of all maximal left ideals of R (so J is a left ideal of R).

a) Prove that if $a \in J$ then $1 + a$ has a left inverse u and $1 - u \in J$. Conclude that $1 + a$ is invertible.

b) Prove that if $r \in R$ then the left ideal Jr is contained in every maximal left ideal of R. Conclude that J is a two-sided ideal. Hint: Take a maximal left ideal M and consider the homomorphism $f : R \longrightarrow R/M$ of left R-modules defined by $f(a) = ar + M$. Wha can you say about the kernel of this homomorphism?

c) Prove that J is contained in every maximal right ideal of R. Conclude that J is the inersection of all maximal right ideals of R.

d) The ideal J is called the **Jacobson radical** of R. Let $a \in R$. Prove that the following conditions are equivalent:

1. $a \in J$.
2. $1 - ra$ has a left inverse for all $r \in R$.
3. $1 - ar$ has a right inverse for all $r \in R$.
4. $1 - ras$ is invertible for every $r, s \in R$.
5. $aM = 0$ for every simple left R-module M.
6. $Ma = 0$ for every simple right R-module M.

Hint: Show that a simple left R-module is isomorphic to R/K for some maximal left ideal K.

Problem 2. Let R be a ring and J its Jacobson radical.

a) Suppose that M is a finitely generated left R-module such that $JM = M$. Prove that $M = 0$ Hint: work with a minimal set of generators of M. This result is called **Nakayama's Lemma**.

b) Suppose that N is a submodule of a finitely generated left R-module M such that $M = N + JM$. Prove that $M = N$.

c) Let $f : M \longrightarrow N$ be a homomorphism of feft R-modules. It induces a homomorphism $\overline{f} : M/JM \longrightarrow N/JN$ of R/J-modules. Prove that if N is finitely generated then f is surjective iff $\overline{f}$ is surjective. Prove that if in addition M is finitely generated and f is a split epimorphism then f is an isomorphism iff $\overline{f}$ is an isomorphism.

d) Suppose that P, Q are left R-modules and $g : P/JP \longrightarrow Q/JQ$ is a surjective homomorphism of R/J-modules. Prove that if P is projective then there is a homomorphism $f : P \longrightarrow Q$ such that $\overline{f} = g$. Conclude that if both P, Q are finitely generarated projective left R-modules then P and Q are isomorphic if and only if P/JP and Q/JQ are isomorphic.

e) Let R be a ring such that R/J is a division ring (such rings are called **local**; this means that R has unique maximal left ideal). Prove that every finitely generated projective R-module is free.

Remark: Kaplansky proved that the same holds for all projective modules (not necessarily finitely generated).

Problem 3. Let $0 \longrightarrow F_n \longrightarrow \ldots \longrightarrow F_0 \longrightarrow P \longrightarrow 0$ be an exact sequence of finitely generated R-modules, where P is projective and F_i are free. Prove that $P \oplus R^m$ is free for some m (such P are called **stably free**).

Problem 4. Let R be a UFD. Prove that an ideal of R is projective iff it is principal.

Problem 5. Let R be a commutative ring such that every finitely generated projective R-module is free. Prove that $(a_1, ..., a_n) \in R^n$ is a row of an invertible $n \times n$ matrix over R iff the ideal generated by $a_1, ..., a_n$ is R. Hint: Note that the condition is equivalent to $(a_1, ..., a_n)$ being part of a basis of R^n. Pick $x_1, ..., x_n$ such that $x_1a_1 + \ldots x_na_n = 1$ and consider the homomorphism $R^n \longrightarrow R$ sending $(b_1, \ldots, b_n)$ to $x_1b_1 + \ldots x_nb_n$.

Problem 6. A left R-module M is called hereditary if every submodule of M is projective. Prove that a direct sum of hereditary R-modules is hereditary.

Problem 7. Let R be a commutative ring.

a) Prove that if R is hereditary and it is not a domain then $R = eR \oplus (1-e)R$ for some non-trivial idempotent e. Prove that each eR and $(1-e)R$ is a hereditary ring.

b) Prove that if $R = S \times T$ is a product of 2 rings then R is hereditary iff both S and T are hereditary.

c) Prove that if R is Noetherian then it is hereditary iff it is a product of finitely many Dedekind domains.

Problem 8. a) Let M and N be left R-modules. Show that there is a well defined natural homomorphism $h_{M,N} : Hom_R(M,R) \otimes_R N \longrightarrow Hom_R(M,N)$ such that $h_{M,N}(f \otimes n)(m) = f(m)n$.

b) Prove that if the identity is in the image of $h_{M,M}$ then M is finitely generated and projective.

c) Prove that if M is finitely generated and projective then $h_{M,N}$ is an isomorphism for every N. Hint: show that $M_1 \oplus M_2$ has this property iff each M_1 and M_2 have it.