Homework 7

due on Wednesday, November 29

Problem 1. Let R be a commutative ring and S a multiplicative subset of R. Show that for any R-module M we have natural isomorphism $S^{-1}M \simeq S^{-1}R \otimes_R M$. Prove that $S^{-1}R$ is a flat R-module.

Problem 2. Let R be a ring with a stricltу increasing chain of left ideals $J_1 \subsetneq J_2 \subsetneq \ldots$. Let $J = \bigcup_{i=0}^{\infty} J_i$. Let M_i be an injective left R-module containing J/J_i for $i = 1, 2, \ldots$. Prove that the module $M = \bigoplus_{i=1}^{\infty} M_i$ is not injective. Conclude that if a direct sum of any countable set of injective left R-modules is injective then R is left Noetherian.

Hinf: Define a homomorphism from J to M which can not be lifted to R. Note that any homomorhism from R to M has image contained in a direct sum of finitely many of the M_i's.

Problem 3. Let $R = \mathbb{Z}[x]$. Let $M = \mathbb{Q}(x)/\mathbb{Z}[x]$, where $\mathbb{Q}(x)$ is the field of rational functions with rational coefficients. Prove that M is a divisible $\mathbb{Z}[x]$-module. Consider the ideal $I = (2, x)$ of $\mathbb{Z}[x]$. Prove that there is a $\mathbb{Z}[x]$-module homomorphism $f : I \longrightarrow M$ such that $f(2) = \frac{1}{x} + \mathbb{Z}[x]$ and $f(x) = \frac{x}{2} + \mathbb{Z}[x]$. Use it to prove that M is not injective $\mathbb{Z}[x]$-module.

Problem 4. Let M be a right R-module and P a flat left R-module. Suppose that $0 = m_1 \otimes p_1 + m_2 \otimes p_2 + \ldots + m_k \otimes p_k$ in $M \otimes_R P$. Prove that there exist elements $q_1, \ldots, q_n$ in P and $r_{i,j} \in R$ for $1 \leq i \leq k$, $1 \leq j \leq n$ such that $p_i = \sum_{j=1}^{n} r_{i,j} q_j$ and $\sum_{i=1}^{k} m_i r_{i,j} = 0$ for all j.

Hint. Use the fact that any exact sequence tensored with P is exact. Apply it to $0 \longrightarrow K \longrightarrow R^k \longrightarrow M$ where the last arrow is the map $f(r_1, \ldots, r_k) = \sum_{i=1}^{k} m_i r_i$ and K is the kernel of f.

Problem 5. Let R be a ring and let $J = J(R)$ be the Jacobson radical of R.

a) Prove that $J(M_n(R)) = M_n(J)$. Conclude that if I is an ideal contained in J then $A \in M_n(R)$ is invertible iff its image in $M_n(R/I)$ is invertible.

b) Prove that if R is commutative then $J(R[x]) = N[x]$, where N is the nilradical of R (problem 3 from second homework may be useful).

c) Prove that if R has no non-zero nil ideals (i.e. two sided ideals whose all elements are nilpotent) then $J(R[x]) = \{0\}$. Hint: Consider a non-zero polynomial of lowest degree in $J(R[x])$ and show that the ideal in R generated by its leading coefficient is nil.

d) For a left R module M define $\mathrm{rad}(M)$ to be the intersection of all maximal submodules of M (set $\mathrm{rad}(M) = M$ if M has no maximal submodules). Prove that $JM \subseteq \mathrm{rad}(M)$ for any left R-module M. Prove that the equality holds for projective modules M. Hint: Show that $\mathrm{rad}(M \oplus N) = \mathrm{rad}(M) \oplus \mathrm{rad}(N)$, and, more generally, $\mathrm{rad}(\bigoplus_{i\in I} M_i) = \bigoplus_{i\in I} \mathrm{rad}(M_i)$.