Midterm, Math 525

October 23

Solve the following problems. Clearly state any main result you are using in your solution. Please write carefully and clearly in **complete sentences**. Take pains to explain what you are doing.

Problem 1. Let R be a commutative ring such that for every $a \in R$ there is a natural number $n > 1$ such that $a^n = a$. Prove that every prime ideal in R is maximal. (6 points)

Problem 2. Let $R = \mathbb{Z}[\sqrt{-7}] = \{a + b\sqrt{-7} : a, b \in \mathbb{Z}\}$ (so this ring is a subring of R_{-7}).

a) Find all invertible elements in R. (1 point)

b) Prove that 2, $1 + \sqrt{-7}$, $1 - \sqrt{-7}$ are irreducible in R. Conclude that R is not a UFD. (4 points)

c) Prove that the ideal $I =< 2, 1 + \sqrt{-7} >$ of R is not principal and that it is maximal. Prove that $I^2 = 2I$. Is there an n such that I^n is principal? (5 points)

Problem 3. Prove that the polynomial $f = x^2y^{2017} + x^{2017}y + x^2 - y - 1$ is irreducible in $\mathbb{Q}[x, y]$. Hint: Consider f as a polynomial in $R[y]$, where $R = \mathbb{Q}[x]$. (8 points)

Problem 4. Let R be a ring which contains a left ideal I minimal among all non-zero left ideals. Suppose that $I^2 \neq 0$.

a) Prove that $Ra = I$ for all $a \in I$, $a \neq 0$. (1 point)

b) Prove that if $a \in I$ then either $Ia = I$ or $Ia = \{0\}$. (2 points)

c) Prove that there is $a \in I$ such that $Ia = I$. Prove that for any such a the map $I \longrightarrow I$ given by $x \mapsto xa$ is bijective. (4 points).

d) Let a be as in c). Prove that there is an idempotent $e \in I$ (i.e. $e^2 = e$) such that $ea = a$. Conclude that $I = Re$. (3 points)

Problem 5. Let R be a PID and let I, J be proper ideals of R.

a) Prove that the intersection of all the ideals I^n, $n = 1, 2, \ldots$, is trivial (this is true, but harder to prove, for any Noetherian integral domain and any ideal I). (2 points)

b) Prove that if $J \neq \{0\}$ then $\bigcap_{n=1}^{\infty}(J + I^n) = J + I^k$ for some k. (6 points)

Problem 6. Let I be an ideal in a commutative ring R and let P_1, P_2, P_2 be prime ideals of R such that $I \subseteq P_1 \cup P_2 \cup P_3$. Prove that $I \subseteq P_i$ for some i (this is true even if only one of the ideals P_i is prime). (6 points)
Hint. Prove this first for 2 prime ideals. Can you generalize it to n prime ideals?