Midterm, Math 525

October 17; Due October 18, 10PM

Solve the following problems. Clearly state any main result you are using in your solution. Please write carefully and clearly in **complete sentences**. Take pains to explain what you are doing. You are expected to work on this exam alone, without consulting with anyone or using any sources except the textbook, the lectures, the homework, and your notes (violating this expectation will be treated as cheating).

Problem 1. Let R be a commutative ring such that for every $a \in R$ there is a natural number $n > 1$ such that $a^n = a$.

a) Prove that every prime ideal in R is maximal. Hint: What can you say when R is an integral domain? (6 points)

b) Prove that the intersection of all prime ideals of R is trivial. (2 points)

Problem 2. Let $R = \mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} : a, b \in \mathbb{Z}\}$ (so this ring is a subring of R_{-3}).

a) Find all invertible elements in R. (1 point)

b) Prove that 2, $1 + \sqrt{-3}$, $1 - \sqrt{-3}$ are irreducible in R. Conclude that R is not a UFD. (3 points)

c) Prove that the ideal $I =< 2, 1 + \sqrt{-3} >$ of R is not principal and that it is maximal. Prove that $I^2 = 2I$. Is there an n such that I^n is principal? (4 points)

Problem 3. Prove that the polynomial $f = x^2y^{2017} + x^{2017}y + x^2 - y - 1$ is a prime element in the ring $\mathbb{Q}[x, y]$. Hint: Consider f as a polynomial in $R[y]$, where $R = \mathbb{Q}[x]$. A homework problem may be useful. (8 points)

Problem 4. Let R be a PID and let I, J be proper ideals of R.

a) Prove that the intersection of all the ideals I^n, $n = 1, 2, \ldots$, is trivial (this is true, but much harder to prove, for any Noetherian integral domain and any ideal I). (2 points)

b) Prove that if $J \neq \{0\}$ then $\bigcap_{n=1}^{\infty}(J + I^n) = J + I^k$ for some k. (6 points)

Problem 5. Let R be a commutative ring and $I =< a, b >$ be an ideal of R generated by two elements a, b and such that $I^2 = I$.

a) Show that every element of I is of the form $ia + jb$ for some $i, j \in I$ (2 points).

b) Suppose that $p, q, s, t \in R$ are such that $pa + qb = 0$ and $sa + tb = 0$. Show that $(pt - sq)a = 0 = (pt - sq)b$ (one way to approach it is by using 2×2 matrices). (2 points)

c) Use a) and b) to show that there is $e \in I$ such that $(1 - e)a = 0 = (1 - e)b$ (3 points).

d) Show that $e^2 = e$ and $I = Re$ (hint: what is $(1 - e)I$?). Conclude that I is a unital ring and $J = R(1 - e)$ is also a unital ring and $R = I \oplus J$. (3 points)

e) (Optional for extra credit) Prove c) when you only know that I is finitely generated.

Problem 6. Let R be a commutative ring.

a) Let $a \in R$ and let M be an ideal of R. Show that the set $J = \{r \in R : ra \in M\}$ is an ideal containing M. (2 points)

b) Let $\mathcal{F}$ be the set of all ideals of R which are not finitely generated. Suppose that $\mathcal{F}$ is not empty. Prove that it contains maximal elements (with respect to inclusion). (3 points)

c) Let M be a maximal element of $\mathcal{F}$ and let $a \notin M$. Show that $M = N + Ja$ for some finitely generated ideal N contained in M, where J is the ideal from part a). Hint: what can you say about the ideal $M + Ra$? Conclude that $J = M$. Conclude that M is a prime ideal. (5 points)

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Extra credit. This problem is optional.

Problem 7. Let $R = M_n(K)$ be the ring of all $n \times n$ matrices with entries in a field K, $n \geq 2$. Let $V = K^n$. We consider elements of V as $1 \times n$ matrices and define $v \cdot A$ to be matrix multiplication ($v \in V$ and $A \in R$).

a) Show that this multiplication makes V a right R-module. Prove that V is a simple R-module..

b) Find two different right ideals I, J of R such that V is isomorphic to each R/I and R/J (consult our discussion of simple modules in the last 8 minutes of lecture 13).

c) Show that if S is a commutative ring and I, J are ideals of S such that the R-modules R/I and R/J are isomorphic then $I = J$.

d) Prove that the right R-modules R and V^n are isomorphic.

e) Conclude that every simple R-module is isomorphic to V.