Homework 4
due on Tuesday, November 3

Read sections 10.1-10.5 in Dummit and Foote. Solve problems 16, 17, 22 to section 10.3 and problems 17,20,21 to section 10.4 of Dummit and Foote. Also solve the following problems:

Problem 1. Let n be a positive integer. Prove that R has IBN (invariant basis number) iff $\mathrm{M}_n(R)$ has IBN.

Problem 2. Consider the ring $C[0,1]$ of all continuous real-valued functions on the interval $[0,1]$. Let R be the subset of $C[0,1]$ which consits of all functions f such that $f(0) = f(1)$ and let M be the subset of $C[0,1]$ which consits of all functions f such that $f(0) = -f(1)$.

a) Prove that R is a subring of $C[0,1]$ and M is an R-module (where the addition and multiplication comes from addition and multiplication in $C[0,1]$).

b) Prove that $M \oplus M$ and $R \oplus R$ are isomorphic as R-modules. Hint: Find an invertible 2 by 2 matrix whose entries are in M.

c) Prove that M is not a free R-module. Hint: Prove that if it was free, it would be isomorhic to R and then derive a contradiction.

Problem 3. For a prime ideal P of a commutative ring R and an R-module M define $M_P = S^{-1}M$, where $S = R - P$. M_P is called the **localization** of M at P. Let $f : M \longrightarrow N$ be a homomorphism of R-modules. Prove that the following are equivalent:

1. f is injective;
2. $\hat{f} : M_P \longrightarrow N_P$ is injective for all prime ideals P;
3. $\hat{f} : M_P \longrightarrow N_P$ is injective for all maximal ideals P;

Prove the same with *injective* replaced by *surjective*.