Homework 5

due on Tuesday, November 17

Read Chapter 10 of Dummit and Foote. Solve problems 14, 27, 28 to 10.5. Also solve the following problems.

Problem 1. Let $0 \longrightarrow F_n \longrightarrow \ldots \longrightarrow F_0 \longrightarrow P \longrightarrow 0$ be an exact sequence of finitely generated R-modules, where P is projective and F_i are free. Prove that $P \oplus R^m$ is free for some m (such P are called **stably free**). Hint: Induction is your friend here.

Problem 2. Let R be a UFD. Prove that an ideal of R is projective iff it is principal. (In class we discussed when an ideal of an integral domain is projective).

Problem 3. Let R be a commutative ring such that every finitely generated projective R-module is free. Prove that $(a_1, ..., a_n) \in R^n$ is a row of an invertible $n \times n$ matrix over R iff $(a_1, ..., a_n) = R$. Hint: Note that the condition is equivalent to $(a_1, ..., a_n)$ being part of a basis of R^n. Pick $x_1, ..., x_n$ such that $x_1a_1 + \ldots + x_na_n = 1$ and consider the homomorphism $R^n \longrightarrow R$ sending $(b_1, \ldots, b_n)$ to $x_1b_1 + \ldots + x_nb_n$.

Problem 4. A left R-module M is called hereditary if every submodule of M is projective. Prove that a direct sum of hereditary R-modules is hereditary. Hint: Follow the proof of Kaplansky's Theorem.

Problem 5. Let R be a commutative ring.

a) Prove that if $R = S \times T$ is a product of 2 rings then R is hereditary iff both S and T are hereditary.

b) Prove that if R is hereditary and it is not a domain then $R = eR \times (1 - e)R$ for some non-trivial idempotent e. Prove that each eR and $(1 - e)R$ is a hereditary ring. Hint: Take a zero divisor a and consider the map $R \longrightarrow Ra$ sending r to ra.

c) Prove that if R is Noetherian then it is hereditary iff it is a product of finitely many Dedekind domains.

Problem 6. a) Let M and N be left R-modules. Show that there is a well defined natural homomorphism $h_{M,N} : Hom_R(M, R) \otimes_R N \longrightarrow Hom_R(M, N)$ such that $h_{M,N}(f \otimes n)(m) = f(m)n$. Hint: Start by defining an R-balance bilinear map $Hom_R(M, R) \times N \longrightarrow Hom_R(M, N)$.

b) Prove that if the identity is in the image of $h_{M,M}$ then M is finitely generated and projective.

c) Prove that if M is finitely generated and projective then $h_{M,N}$ is an isomorphism for every N. Hint: show that $M_1 \oplus M_2$ has this property iff each M_1 and M_2 have it.

Problem 7. Let R be a Dedekind domain and S a multiplicative subset of R. Prove that $S^{-1}R$ is a Dedekind domain.

Problem 8. Let R be a ring with a striclty increasing chain of right ideals $J_1 \subsetneq J_2 \subsetneq \ldots$. Let $J = \bigcup_{i=0}^{\infty} J_i$. Let M_i be an injective right R-module containing J/J_i for $i = 1, 2, \ldots$. Prove that the module $M = \bigoplus_{i=1}^{\infty} M_i$ is not injective. Conclude that if a direct sum of any countable set of injective right R-modules is injective then R is right Noetherian.

Hint: Define a homomorphism from J to M which can not be lifted to R. In your argument the following observation should be useful: any homomorhism from R to M has image contained in a direct sum of finitely many of the M_i's.

Problem 9. Let $R = \mathbb{Z}[x]$. Let $M = \mathbb{Q}(x)/\mathbb{Z}[x]$, where $\mathbb{Q}(x)$ is the field of rational functions with rational coefficients. Prove that M is a divisible $\mathbb{Z}[x]$-module. Consider the ideal $I = (2, x)$ of $\mathbb{Z}[x]$. Prove that there is a $\mathbb{Z}[x]$-module homomorphism $f : I \longrightarrow M$ such that $f(2) = \frac{1}{x} + \mathbb{Z}[x]$ and $f(x) = \frac{x}{2} + \mathbb{Z}[x]$. Use it to prove that M is not injective $\mathbb{Z}[x]$-module.