Homework 6
due on Thursday, December 3

Read carefully chapter 12 of Dummit and Foote. Solve problems 11, 37, 38 to section 12.3. Also solve the following problems.

**Problem 1.** Let $R$ be a ring. Let $J$ be the intersection of all maximal left ideals of $R$ (so $J$ is a left ideal of $R$).

a) Prove that if $a \in J$ then $1 - a$ has a left inverse $u$ and $1 - u \in J$. Conclude that $1 - a$ is invertible.

b) Prove that if $r \in R$ then the left ideal $Jr$ is contained in every maximal left ideal of $R$. Conclude that $J$ is a two-sided ideal. Hint: Take a maximal left ideal $M$ and consider the homomorphism $f : R \rightarrow R/M$ of left $R$-modules defined by $f(a) = ar + M$. Note that $R/M$ is a simple left $R$-module. What can you say about the kernel of this homomorphism?

Alternatively, assume $jr + m = 1$ for some $j \in J$ and $m \in M$ and conclude that $r = (1 - rj)^{-1}rm \in M$, a clear contradiction.

c) Prove that $J$ is contained in every maximal right ideal of $R$. Conclude that $J$ is the intersection of all maximal right ideals of $R$ (reverse the roles of left and right).

d) The ideal $J$ is called the **Jacobson radical** of $R$. Let $a \in R$. Prove that the following conditions are equivalent:

1. $a \in J$.

2. $1 - ra$ has a left inverse for all $r \in R$.

3. $1 - ar$ has a right inverse for all $r \in R$.

4. $1 - ras$ is invertible for every $r, s \in R$.

5. $aM = 0$ for every simple left $R$-module $M$.

6. $Ma = 0$ for every simple right $R$-module $M$.

**Hint:** Recall that a left $R$-module is simple iff it isomorphic to $R/K$ for some maximal left ideal $K$. 

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Problem 2. Let $R$ be a ring and $J$ its Jacobson radical.

a) Suppose that $M$ is a finitely generated left $R$-module such that $JM = M$. Prove that $M = 0$. Hint: work with a minimal set of generators of $M$. This result is called Nakayama’s Lemma.

b) Suppose that $N$ is a submodule of a finitely generated left $R$-module $M$ such that $M = N + JM$. Prove that $M = N$.

c) Let $f : M \rightarrow N$ be a homomorphism of left $R$-modules. It induces a homomorphism $\bar{f} : M/JM \rightarrow N/JN$ of $R/J$-modules. Prove that if $N$ is finitely generated then $f$ is surjective iff $\bar{f}$ is surjective. Prove that if in addition $M$ is finitely generated and $f$ is a split epimorphism then $f$ is an isomorphism iff $\bar{f}$ is an isomorphism.

d) Suppose that $P, Q$ are left $R$-modules and $g : P/JP \rightarrow Q/JQ$ is a surjective homomorphism of $R/J$-modules. Prove that if $P$ is projective then there is a homomorphism $f : P \rightarrow Q$ such that $\bar{f} = g$. Conclude that if both $P, Q$ are finitely generated projective left $R$-modules then $P$ and $Q$ are isomorphic if and only if $P/JP$ and $Q/JQ$ are isomorphic.

e) Let $R$ be a ring such that $R/J$ is a division ring (such rings are called local; this means that $R$ has unique maximal left ideal). Prove that every finitely generated projective $R$-module is free.

Remark: Kaplansky proved that the same holds for all projective modules (not necessarily finitely generated).

Problem 3. Let $R$ be a ring and let $J = J(R)$ be the Jacobson radical of $R$.

a) Prove that $J(M_n(R)) = M_n(J)$. Conclude that if $I$ is an ideal contained in $J$ then $A \in M_n(R)$ is invertible iff its image in $M_n(R/I)$ is invertible.

b) Prove that if $R$ is commutative then $J(R[x]) = N[x]$, where $N$ is the nilradical of $R$ (problem 6 from third homework may be useful).

c) Prove that if $R$ has no non-zero nil ideals (i.e. two sided ideals whose all elements are nilpotent) then $J(R[x]) = \{0\}$. Hint: Consider a non-zero polynomial of lowest degree in $J(R[x])$ and show that its leading coefficient commutes with all the other
coefficients. Use this to show that the ideal in \( R \) generated by the leading leading coefficient is nil.

d) For a left \( R \) module \( M \) define \( \text{rad}(M) \) to be the intersection of all maximal submodules of \( M \) (set \( \text{rad}(M) = M \) if \( M \) has no maximal submodules). Prove that \( JM \subseteq \text{rad}(M) \) for any left \( R \)-module \( M \). Prove that the equality holds for projective modules \( M \). Hint: Show that \( \text{rad}(M \oplus N) = \text{rad}(M) \oplus \text{rad}(N) \), and, more generally, \( \text{rad}(\bigoplus_{i \in I} M_i) = \bigoplus_{i \in I} \text{rad}(M_i) \).

Problem 4. Let \( M \) be a left \( R \)-module and \( f \) an endomorphism of \( M \). For each \( n \) let \( K_n, I_n \) be the kernel and image of \( f^n \) respectively.

a) Prove that if \( I_n = I_{n+1} \) then \( M = I_n + K_n \).

b) Prove that if \( K_n = K_{n+1} \) then \( I_n \cap K_n = \{0\} \).

c) Prove that if \( M \) is Artinian then \( M = I_n + K_n \) for all sufficiently large \( n \).

d) Prove that if \( M \) is Noetherian then \( I_n \cap K_n = \{0\} \) for all sufficiently large \( n \).

e) Prove that if \( M \) is Artinian and \( f \) is a monomorphism then \( f \) is an isomorphism.

f) Prove that if \( M \) is Noetherian and \( f \) is surjective then \( f \) is an isomorphism.

(Remark: If \( R \) is commutative this is true for any finitely generated module \( M \). Can you prove it?)

g) Suppose that \( M \) has finite length and cannot be decomposed into a direct sum of proper submodules. Prove that either \( f \) is nilpotent or it is an isomorphism. Conclude that in the ring \( \text{End}_R(M) \) the Jacobson radical \( J \) consists of nilpotent elements and \( \text{End}_R(M)/J \) is a division ring.

Problem 5. a) Let \( F \) be a free right \( R \)-module with basis \( e_1, \ldots, e_n \) and let \( P \) be a left \( R \)-module. Show that every element of \( F \otimes_R P \) can be written in a unique way as \( e_1 \otimes p_1 + \ldots + e_n \otimes p_n \) for some \( p_1, \ldots, p_n \in P \).

b) Let \( M \) be a right \( R \)-module and \( P \) a flat left \( R \)-module. Suppose that \( 0 = m_1 \otimes p_1 + m_2 \otimes p_2 + \ldots + m_k \otimes p_k \) in \( M \otimes_R P \). Prove that there exist elements \( q_1, \ldots, q_n \) in \( P \) and \( r_{i,j} \in R \) for \( 1 \leq i \leq k, 1 \leq j \leq n \) such that \( p_i = \sum_{j=1}^n r_{i,j} q_j \) and \( \sum_{i=1}^k m_i r_{i,j} = 0 \) for all \( j \).
Hint. Use the fact that any exact sequence tensored with $P$ is exact. Apply it to $0 \rightarrow K \rightarrow R^k \rightarrow M$ where the last arrow is the map $f(r_1, \ldots, r_k) = \sum_{i=1}^{k} m_i r_i$ and $K$ is the kernel of $f$.

c) Let $P$ be a left $R$-module such that for any $r_1, \ldots, r_k \in R$ and any $p_1, \ldots, p_k \in P$ such that $r_1 p_1 + r_2 p_2 + \ldots + r_k p_k = 0$ there exist elements $q_1, \ldots, q_n$ in $P$ and $r_{i,j} \in R$ for $1 \leq i \leq k, 1 \leq j \leq n$ such that $p_i = \sum_{j=1}^{n} r_{i,j} q_j$ and $\sum_{i=1}^{k} r_i r_{i,j} = 0$ for all $j$. Prove that $P$ is flat. Hint: Use "Baer’s Criterion" for flat modules.