Homework 6

due on Thursday, December 3

Read carefully chapter 12 of Dummit and Foote. Solve problems 11,37,38 to section 12.3. Also solve the following problems.

Problem 1. Let R be a ring. Let J be the intersection of all maximal left ideals of R (so J is a left ideal of R).

a) Prove that if $a \in J$ then $1-a$ has a left inverse u and $1-u \in J$. Conclude that $1-a$ is invertible.

b) Prove that if $r \in R$ then the left ideal Jr is contained in every maximal left ideal of R. Conclude that J is a two-sided ideal. Hint: Take a maximal left ideal M and consider the homomorphism $f : R \longrightarrow R/M$ of left R-modules defined by $f(a) = ar + M$. Note that R/M is a simple left R-module. What can you say about the kernel of this homomorphism?

Alternatively, assume $jr + m = 1$ for some $j \in J$ and $m \in M$ and conclude that $r = (1-rj)^{-1}rm \in M$, a clear contradiction.

c) Prove that J is contained in every maximal right ideal of R. Conclude that J is the inersection of all maximal right ideals of R (reverse the roles of left and right).

d) The ideal J is called the **Jacobson radical** of R. Let $a \in R$. Prove that the following conditions are equivalent:

1. $a \in J$.
2. $1 - ra$ has a left inverse for all $r \in R$.
3. $1 - ar$ has a right inverse for all $r \in R$.
4. $1 - ras$ is invertible for every $r, s \in R$.
5. $aM = 0$ for every simple left R-module M.
6. $Ma = 0$ for every simple right R-module M.

Hint: Recall that a left R-module is simple iff it isomorphic to R/K for some maximal left ideal K.

Problem 2. Let R be a ring and J its Jacobson radical.

a) Suppose that M is a finitely generated left R-module such that $JM = M$. Prove that $M = 0$ Hint: work with a minimal set of generators of M. This result is called **Nakayama's Lemma**.

b) Suppose that N is a submodule of a finitely generated left R-module M such that $M = N + JM$. Prove that $M = N$.

c) Let $f : M \longrightarrow N$ be a homomorphism of feft R-modules. It induces a homomorphism $\overline{f} : M/JM \longrightarrow N/JN$ of R/J-modules. Prove that if N is finitely generated then f is surjective iff $\overline{f}$ is surjective. Prove that if in addition M is finitely generated and f is a split epimorphism then f is an isomorphism iff $\overline{f}$ is an isomorphism.

d) Suppose that P, Q are left R-modules and $g : P/JP \longrightarrow Q/JQ$ is a surjective homomorphism of R/J-modules. Prove that if P is projective then there is a homomorphism $f : P \longrightarrow Q$ such that $\overline{f} = g$. Conclude that if both P, Q are finitely generarated projective left R-modules then P and Q are isomorphic if and only if P/JP and Q/JQ are isomorphic.

e) Let R be a ring such that R/J is a division ring (such rings are called **local**; this means that R has unique maximal left ideal). Prove that every finitely generated projective R-module is free.
Remark: Kaplansky proved that the same holds for all projective modules (not necessarily finitely generated).

Problem 3. Let R be a ring and let $J = J(R)$ be the Jacobson radical of R.

a) Prove that $J(M_n(R)) = M_n(J)$. Conclude that if I is an ideal contained in J then $A \in M_n(R)$ is invertible iff its image in $M_n(R/I)$ is invertible.

b) Prove that if R is commutative then $J(R[x]) = N[x]$, where N is the nilradical of R (problem 6 from third homework may be useful).

c) Prove that if R has no non-zero nil ideals (i.e. two sided ideals whose all elements are nilpotent) then $J(R[x]) = \{0\}$. Hint: Consider a non-zero polynomial of lowest degree in $J(R[x])$ and show that its leading coefficient commutes with all the other coefficients. Use this to show that the ideal in R generated by the leading leading coefficient is nil.

d) For a left R module M define $\mathrm{rad}(M)$ to be the intersection of all maximal submodules of M (set $\mathrm{rad}(M) = M$ if M has no maximal submodules). Prove that $JM \subseteq \mathrm{rad}(M)$ for any left R-module M. Prove that the equality holds for projective modules M. Hint: Show that $\mathrm{rad}(M \oplus N) = \mathrm{rad}(M) \oplus \mathrm{rad}(N)$, and, more generally, $\mathrm{rad}(\bigoplus_{i\in I} M_i) = \bigoplus_{i\in I} \mathrm{rad}(M_i)$.

Problem 4. Let M be a left R-module and f and endomorphism of M. For each n let K_n, I_n be the kernel and image of f^n respectively.

a) Prove that if $I_n = I_{n+1}$ then $M = I_n + K_n$.

b) Prove that if $K_n = K_{n+1}$ then $I_n \cap K_n = \{0\}$.

c) Prove that if M is Artinian then $M = I_n + K_n$ for all sufficiently large n.

d) Prove that if M is Noetherian then $I_n \cap K_n = \{0\}$ for all sufficiently large n.

e) Prove that if M is Artinian and f is a monomorphism then f is an isomorphism.

f) Prove that if M is Noetherian and f is surjective then f is an isomorphism. (Remark: If R is commutative this is true for any finitely generated module M. Can you prove it?)

g) Suppose that M has finite length and cannot be decomposed into a direct sum of proper submodules. Prove that either f is nilpotent or it is an isomorphism. Conclude that in the ring $\mathrm{End}_R(M)$ the Jacobson radical J consists of nilpotent elements and $\mathrm{End}_R(M)/J$ is a division ring.

Problem 5. a) Let F be a free right R-module with basis $e_1, \ldots, e_n$ and let P be a left R-module. Show that every element of $F \otimes_R P$ can be written in a unique way as $e_1 \otimes p_1 + \ldots + e_n \otimes p_n$ for some $p_1, \ldots, p_n \in P$.

b) Let M be a right R-module and P a flat left R-module. Suppose that $0 = m_1 \otimes p_1 + m_2 \otimes p_2 + \ldots + m_k \otimes p_k$ in $M \otimes_R P$. Prove that there exist elements $q_1, \ldots, q_n$ in P and $r_{i,j} \in R$ for $1 \le i \le k$, $1 \le j \le n$ such that $p_i = \sum_{j=1}^{n} r_{i,j} q_j$ and $\sum_{i=1}^{k} m_i r_{i,j} = 0$ for all j.

Hint. Use the fact that any exact sequence tensored with P is exact. Apply it to $0 \longrightarrow K \longrightarrow R^k \longrightarrow M$ where the last arrow is the map $f(r_1, \ldots, r_k) = \sum_{i=1}^k m_i r_i$ and K is the kernel of f.

c) Let P be a left R-module such that for any $r_1, \ldots, r_k \in R$ and any $p_1, \ldots, p_k \in P$ such that $r_1 p_1 + r_2 p_2 + \ldots + r_k p_k = 0$ there exist elements $q_1, \ldots, q_n$ in P and $r_{i,j} \in R$ for $1 \leq i \leq k$, $1 \leq j \leq n$ such that $p_i = \sum_{j=1}^n r_{i,j} q_j$ and $\sum_{i=1}^k r_i r_{i,j} = 0$ for all j. Prove that P is flat. Hint: Use "Baer's Criterion" for flat modules.