**Homework 1**
due on Monday, September 9

Study chapter 7 of Dummit and Foote, go through as many excercises as possible. Solve problems 33 and 34 to 7.4 and the following problems.

**Problem 1.** An element $a$ of a ring $R$ is called **nilpotent** if $a^m = 0$ for some $m > 0$.

a) Prove that in a commutative ring $R$ the set $N$ of all nilpotent elements of $R$ is an ideal. This ideal is called the **nilradical** of $R$. Prove that 0 is the only nilpotent element of $R/N$.

b) Let $R$ be a commutative ring and let $a_1, ..., a_n \in R$ be nilpotent. Set $I$ for the ideal $< a_1, ..., a_n >$ generated by $a_1, ..., a_n$. Prove that there is a positive integer $N$ such that $x_1 x_2 ... x_N = 0$ for any $x_1, ..., x_N$ in $I$ (i.e. that $I^N = 0$).

c) Prove that the set of all nilpotent elements in the ring $M_2(\mathbb{R})$ is not an ideal.

d) Prove that if $p$ is a prime and $m > 0$ then every element of $\mathbb{Z}/p^m\mathbb{Z}$ is either nilpotent or invertible.

e) Find the nilradical of $\mathbb{Z}/36\mathbb{Z}$ (by correspondence theorem, it is equal to $n\mathbb{Z}/36\mathbb{Z}$ for some $n$).

**Problem 2.** Let $R$ be a commutative ring. For an ideal $I$ of $R$ define

$$\sqrt{I} = \{x \in R : x^n \in I \text{ for some } n > 0\}.$$

a) Prove that $\sqrt{I}$ is an ideal. It is called the **radical** of $I$.

b) Prove that $\sqrt{\{0\}}$ is the nilradical of $R$.

c) Consider a surjective homomorphism $f : R \longrightarrow S$. Prove that in the correspondence theorem the nilradical of $S$ corresponds to $\sqrt{\ker f}$.

d) Prove that $R/\sqrt{I}$ has trivial nilradical.

**Problem 3.** A subset $S$ of a commutative ring is called **multiplicative** if $0 \notin S$ and for any $a, b \in S$ also $ab \in S$. .

a) Let $I$ be an ideal of a commutatuve unital ring $R$. Prove that $I$ is a prime ideal iff $R - I$ is multiplicative.

b) Let $S$ be a multiplicative subset of a comutative unital ring $R$. Consider the set $T$ of all ideals of $R$ which are disjoint with $S$. Prove that this set contains maximal elements (with respect to inclusion; this requires Zorn's Lemma and is very similar to the proof that every ring has a maximal ideal). Prove that every maximal element of $T$ is a prime ideal.

c) Use b) to prove that if $a \in R$ is not nilpotent then there is a prime ideal in $R$ which does not contain $a$.

d) Prove that the nilradical of a commutative unital ring $R$ coincides with the intersection of all prime ideals.

**Problem 4.** Let $f : R \longrightarrow S$ be a homomorphism of commutative unital rings.

a) Prove that if $P$ is a prime ideal of $S$ then $f^{-1}(P)$ is a prime ideal of $R$. Is this true for non-commutative rings?

b) Find an example when $P$ is a maximal ideal of $S$ but $f^{-1}(P)$ is not maximal in $R$.

c) Prove that if $f$ is onto and $Q$ is a prime ideal of $R$ such that $\ker f \subseteq Q$ then $f(Q)$ is a prime ideal of $S$. Is this true for non-commutative rings?

d) Suppose that $f$ is surjective. Prove that if $P$ is a maximal ideal of $S$ then $f^{-1}(P)$ is maximal in $R$. Prove that if $Q$ is a maximal ideal of $R$ then $f(Q)$ is either $S$ or it is a maximal ideal of $S$. Show by example that a similar statement for prime ideals is false.

e) Find all prime ideals of $\mathbb{Z}/36\mathbb{Z}$.

**Problem 5.** Let $R$ be a ring and $n$ a positive integer. Prove that if $I$ is an ideal of the ring $\mathrm{M}_n(R)$ then $I = \mathrm{M}_n(J)$ for some ideal $J$ of $R$. Prove that $I$ is maximal iff $J$ is maximal. Prove that $I$ is prime iff $J$ is prime. Conclude that if $R$ is simple (prime) then so is $\mathrm{M}_n(R)$.

**Problem 6.** Let $I$ be a prime ideal of $R$. Prove that if $J$, $K$ are left ideals of $R$

such that $JK \subseteq I$ then either $J \subseteq I$ or $K \subseteq I$. Hint: Consider the set of all $a$ such that $aK \subseteq I$ and prove that it is an ideal containing $J$.