Homework 3

due on Monday, October 7

Study Chapters 7, 8 in Dummit and Foote. Solve the following problems.

Problem 1. Solve problems 3,4 to section 7.2 in Dummit and Foote. In addition, prove that when R is a field, then $R[[x]]$ is an Euclidean domain. Consult problem 5 to 7.2. and example 4 in section 8.1.

Problem 2. Find a greatest common divisor $d(x)$ of the polynomials $p(x) = x^3 + 4x^2 + x - 6$ and $q(x) = x^5 - 6x + 5$ in the ring $\mathbb{Q}[x]$ and find $a(x), b(x) \in \mathbb{Q}[x]$ such that $d(x) = a(x)p(x) + b(x)q(x)$.

Problem 3. Let $K \subseteq L$ be fields. Suppose that $f, g \in K[x]$ and $f|g$ in the ring $L[x]$. Prove that $f|g$ in the ring $K[x]$.

Problem 4. Ler R be an integral domain.

a) Let $f, g \in R[x]$ be such that $fg = cx^n$ for some n and some $c \in R$, $c \neq 0$. Prove that there exist elements $a, b \in R$ and $m \leq n$ such that $f = ax^m$ and $g = bx^{n-m}$ and $ab = c$.

b) Suppose that $f = f_0 + f_1x + ... + f_nx^n \in R[x]$. Suppose that there is a prime ideal P of R such that $f_n \notin P$, $f_0, ..., f_{n-1} \in P$ and $f_0 \notin P^2$. Prove that if $f = gh$ for some $g, h \in R[x]$ then one of g, h is constant. Conclude that if in addition f is monic then it is irreducible in $R[x]$. This result is known as **Eisenstein criterion**. Hint: Assume that $f = gh$ and both g, h have positive degree. Pass to the ring $(R/P)[x]$ and apply a) to show that constant terms of g and h belong to P. Derive contradiction.

c) Prove that the polynomial $2x^{10} + 21x^8 - 35x^2 + 14$ is irreducible in $\mathbb{Z}[x]$. Hint: Apply Eisenstein criterion with appropriate prime ideal P.

Problem 5. Let R be an integral domain with ACCP. Prove that $R[x]$ has ACCP.

Problem 6. Let R be a commutative ring and let $R[x]$ be the ring of polynomials in x with coefficients in R. Let $f = f_0 + f_1x + ... + f_nx^n \in R[x]$. Prove that

a) f is invertible iff $f_0 \in R^\times$ and $f_1, ..., f_n$ are nilpotent.

b) f is nilpotent iff $f_0, ..., f_n$ are nilpotent.

c) f is a zero divisor iff $af = 0$ for some $0 \neq a \in R$.

d) Let P be a prime ideal of R and $f, g \in R[x]$. Prove that all coefficients of fg belong to P iff either all coefficients of f or all coefficients of g belong to P.

e) If f belongs to every maxiam ideal of $R[x]$ then f is nilpotent.

Problem 7. Let F be a finite field with $q = p^n$ elements. Let a be a generator of the multiplicative group $F^\times$ (we proved that this group is cyclic).

a) Show that there is unique monic irreducible polynomial $g \in \mathbb{F}_p[x]$ of degree n such that $g(a) = 0$. Prove that if b is a root of g then so is b^p. Conclude that $a, a^p, ..., a^{p^{n-1}}$ are distinct roots of g.

b) Let ϕ be an automorphism of the field F. Prove that there is $0 \leq k < n$ such that $\phi(x) = x^{p^k}$. Conclude that the group of all automorphisms of F is cyclic of order n.

c) Let I_n be the set of all monic irreducible polynomials of degree n in $\mathbb{F}_p[x]$. Prove that

$$x^{p^n} - x = \prod_{k|n} \prod_{f \in I_k} f.$$

Let i_n be the cardinality of I_n. Conclude that

$$p^n = \sum_{k|n} k i_k.$$

This allows to compute i_k for every k.