Homework 4

due on Friday, October 18

Study Chapters 7, 8, 9 in Dummit and Foote. Solve the following problems.

Problem 1. Let A be an ordered abelian group (like the integers). A valuation on an integral domain R is a function $v : R - \{0\} \longrightarrow A$ such that

1. $v(ab) = v(a) + v(b)$ for all $a, b \in A$;
2. $v(a + b) \geq \min(v(a), v(b))$ for all $a, b \in A,$ such that $a + b \neq 0$

Let v be a valuation on R.

a) Let K be the field of fractions of R. For a non-zero element a/b of Kdefine $v(a/b) = v(a) - v(b)$. Prove that v is well defined and it is a valuation

b) Define a function $w : R[x] - \{0\} \longrightarrow A$ by $w(f) =$ the smallest of the valuations of the non-zero coefficients of the polynomial f. Prove that w is a valuation on $R[x]$.

c) Use b) to prove Gauss' Lemma. Hint: if R is a UFD then any irreducible element of R corresponds to a discrete valuation on R (i.e. the valuation has values in the integers; we discussed it in class).

Problem 2. Ler R be a UFD and let S be a multiplicative subset of R. Prove that $S^{-1}R$ is a UFD. Is the same true with UFD replaced by PID?

Problem 3. Let P be a prime ideal of the ring S_d. Prove that the localization of S_d at P is a PID.

Problem 4. Let R be an integral domain and let $a \in R$ be such that R/aR is finite. Show that R/a^nR is finite and $|R/a^nR| = |R/aR|^n$.

Problem 5. Let K be a field and let R be an integral domain containing K as a subring and finite dimensional as a K-vector space. Prove that R is a field.