Homework 5

due on Monday, October 28

Read sections 10.1-10.3 in Dummit and Foote. Solve problem 13 to section 10.2 and problems 16, 17, 22 to section 10.3 of Dummit and Foote. (For problem 17, either assume R is commutative (otherwise the result is not true), or formulate and prove a version for non-commutative rings (following what we did in class for the Chinese Reminder Theorem). Also solve the following problems:

Problem 1. Let R be a commutative ring and S a multiplicative subset of R. For an R module M consider the set $M \times S$ and the relation $(m, s) \sim (n, t)$ iff $r(tm - sn) = 0$ for some $r \in S$.

a) Show that $\sim$ is an equivalence relation. Denote the equivalence class of (m, s) by $\frac{m}{s}$ and the set of all equivalence classes by $S^{-1}M$. Prove that the operation

$$\frac{m}{s} + \frac{n}{t} = \frac{tm + sn}{st}$$

is well defined and makes $S^{-1}M$ and abelian group.

b) For $\frac{r}{t} \in S^{-1}R$ and $\frac{m}{s} \in S^{-1}M$ define

$$\frac{r}{t}\frac{m}{s} = \frac{rm}{ts}.$$

Prove that this is a well defined operation which makes $S^{-1}M$ into and $S^{-1}R$-module.

c) Let $f : M \longrightarrow N$ be a homomorphism of R-modules. Show that $\hat{f} : S^{-1}M \longrightarrow S^{-1}N$ given by

$$\hat{f}(\frac{m}{s}) = \frac{f(m)}{s}$$

is well defined and it is a homomorphism of $S^{-1}R$-modules.

d) A sequence of R-module homomorphisms $M \xrightarrow{f} N \xrightarrow{g} P$ is **exact** if the kernel of g coincides with the image of f. Prove that if

$$M \xrightarrow{f} N \xrightarrow{g} P$$

is an exact sequence then so is

$$S^{-1}M \xrightarrow{\hat{f}} S^{-1}N \xrightarrow{\hat{g}} S^{-1}P.$$

In particular, if M is a submodule of N then $S^{-1}M$ can be naturally considered as a submodule of $S^{-1}N$.

e) Let N, P be submodules of an R-module M. Prove that

1. $S^{-1}(N+P) = S^{-1}N + S^{-1}P$;
2. $S^{-1}(N \cap P) = S^{-1}N \cap S^{-1}P$;
3. the $S^{-1}R$-modules $S^{-1}(M/N)$ and $S^{-1}M/S^{-1}N$ are isomorphic.

f) For an R-module M define the annihilator of M as

$$ann(M) = \{r \in R : rm = 0 \text{ for all } m \in M\}.$$

Prove that $ann(M)$ is an ideal. Prove that $S^{-1}ann(M) = ann(S^{-1}M)$ provided M is a finitely generated R-module.

g) Let N, P be submodules of an R-module M. Define

$$(N : P) = \{r \in R : rx \in N \text{ for all } x \in P\}.$$

Prove that $(N : P)$ is an ideal in R. Prove that $S^{-1}(N : P) = (S^{-1}N : S^{-1}P)$ provided P is finitely generated.

Problem 2. For a prime ideal P of a commutative ring R and an R-module M define $M_P = S^{-1}M$, where $S = R - P$. M_P is called the **localization** of M at P. Let $f : M \longrightarrow N$ be a homomorphism of R-modules. Prove that the following are equivalent:

1. f is injective;
2. $\hat{f} : M_P \longrightarrow N_P$ is injective for all prime ideals P;
3. $\hat{f} : M_P \longrightarrow N_P$ is injective for all maximal ideals P;

Prove the same with *injective* replaced by *surjective*.

Problem 3. Consider the ring $C[0, 1]$ of all continuous real-valued functions on the interval $[0, 1]$. Let R be the subset of $C[0, 1]$ which consits of all functions f such that $f(0) = f(1)$ and let M be the subset of $C[0, 1]$ which consits of all functions f such that $f(0) = -f(1)$.

a) Prove that R is a subring of $C[0, 1]$ and M is an R-module (where the addition and multiplication comes from addition and multiplication in $C[0, 1]$).

b) Prove that $M \oplus M$ and $R \oplus R$ are isomorphic as R-modules. Hint: Find an invertible 2 by 2 matrix whose entries are in M.

c) Prove that M is not a free R-module. Hint: Prove that if it was free, it would be isomorhic to R and then derive a contradiction.

Problem 4. Let n be a positive integer. Prove that R has IBN (invariant basis number) iff $\mathrm{M}_n(R)$ has IBN.