Homework 6

due on Friday, November 15

Read Chapter 10 of Dummit and Foote. Solve problems 17, 20, 21 to section 10.4 and 14, 27, 28 to 10.5. Also solve the following problems.

Problem 1. Let $0 \longrightarrow F_n \longrightarrow \ldots \longrightarrow F_0 \longrightarrow P \longrightarrow 0$ be an exact sequence of finitely generated R-modules, where P is projective and F_i are free. Prove that $P \oplus R^m$ is free for some m (such P are called **stably free**). Hint: Induction is your friend here.

Problem 2. Let R be a commutative ring such that every finitely generated projective R-module is free. Prove that $(a_1, ..., a_n) \in R^n$ is a row of an invertible $n \times n$ matrix over R iff the ideal generated by $a_1, ..., a_n$ is R. Hint: Note that the condition is equivalent to $(a_1, ..., a_n)$ being part of a basis of R^n. Pick $x_1, ..., x_n$ such that $x_1a_1+\ldots+x_na_n = 1$ and consider the homomorphism $R^n \longrightarrow R$ sending $(b_1, \ldots, b_n)$ to $x_1b_1 + \ldots + x_nb_n$. What can you say about the kernel of this homomorphism?

Problem 3. a) Let M and N be left R-modules. Show that there is a well defined natural homomorphism $h_{M,N} : Hom_R(M, R) \otimes_R N \longrightarrow Hom_R(M, N)$ such that $h_{M,N}(f \otimes n)(m) = f(m)n$. Hint: Start by defining an R-balance bilinear map $Hom_R(M, R) \times N \longrightarrow Hom_R(M, N)$.

b) Prove that if the identity is in the image of $h_{M,M}$ then M is finitely generated and projective.

c) Prove that if M is finitely generated and projective then $h_{M,N}$ is an isomorphism for every N. Hint: show that $M_1 \oplus M_2$ has this property iff each M_1 and M_2 have it.