Homework 7
due on Wednesday, December 4

Read carefully section 10.5 and chapter 12 of Dummit and Foote. Solve problems 37,38 to section 12.3 (focus first on the problems below). Also solve the following problems.

Problem 1. Let R be a commutative ring, S a multiplicative set in R.

a) Let M be an R-module. Prove that $S^{-1}M$ is isomorphic to $S^{-1}R \otimes_R M$ (see problems 1,2 of homework 5).

b) Use problem 1 of homework 5 to show that $S^{-1}R$ is flat as R-module.

c) Show that if P is a projective R-module then $S^{-1}P$ is a projective $S^{-1}R$-module

d) Let R be a Dedekind domain. Prove that $S^{-1}R$ is a Dedekind domain.

Problem 2. Let R be a ring. Let J be the intersection of all maximal left ideals of R (so J is a left ideal of R).

a) Prove that if $a \in J$ then $1 - a$ has a left inverse u and $1 - u \in J$. Conclude that $1 - a$ is invertible.

b) Prove that if $r \in R$ then the left ideal Jr is contained in every maximal left ideal of R. Conclude that J is a two-sided ideal. Hint: Take a maximal left ideal M and consider the homomorphism $f : R \longrightarrow R/M$ of left R-modules defined by $f(a) = ar + M$. Note that R/M is a simple left R-module. What can you say about the kernel of this homomorphism (what kind of left ideal is it?)

Alternatively, assume $jr + m = 1$ for some $j \in J$ and $m \in M$ and conclude that $r = (1 - rj)^{-1}rm \in M$, a clear contradiction.

c) Prove that J is contained in every maximal right ideal of R. Conclude that J is the inersection of all maximal right ideals of R (reverse the roles of left and right).

d) The ideal J is called the **Jacobson radical** of R. Let $a \in R$. Prove that the following conditions are equivalent:

1. $a \in J$.
2. $1 - ra$ has a left inverse for all $r \in R$.
3. $1 - ar$ has a right inverse for all $r \in R$.
4. $1 - ras$ is invertible for every $r, s \in R$.
5. $aM = 0$ for every simple left R-module M.
6. $Ma = 0$ for every simple right R-module M.

Hint: Recall that a left R-module is simple iff it isomorphic to R/K for some maximal left ideal K.

Problem 3. Let R be a ring and J its Jacobson radical.

a) Suppose that M is a finitely generated left R-module such that $JM = M$. Prove that $M = 0$ Hint: work with a minimal set of generators of M. This result is called **Nakayama's Lemma**.

b) Suppose that N is a submodule of a finitely generated left R-module M such that $M = N + JM$. Prove that $M = N$.

c) Let $f : M \longrightarrow N$ be a homomorphism of feft R-modules. It induces a homomorphism $\overline{f} : M/JM \longrightarrow N/JN$ of R/J-modules. Prove that if N is finitely generated then f is surjective iff $\overline{f}$ is surjective. Prove that if in addition M is finitely generated and f is a split epimorphism then f is an isomorphism iff $\overline{f}$ is an isomorphism.

d) Suppose that P, Q are left R-modules and $g : P/JP \longrightarrow Q/JQ$ is a surjective homomorphism of R/J-modules. Prove that if P is projective then there is a homomorphism $f : P \longrightarrow Q$ such that $\overline{f} = g$. Conclude that if both P, Q are finitely generarated projective left R-modules then P and Q are isomorphic if and only if P/JP and Q/JQ are isomorphic.

e) Let R be a ring such that R/J is a division ring (such rings are called **local**; this means that R has unique maximal left ideal). Prove that every finitely generated projective R-module is free.

Remark: Kaplansky proved that the same holds for all projective modules (not necessarily finitely generated).

Problem 4. Let R be a ring and let $J = J(R)$ be the Jacobson radical of R.

a) Prove that $J(M_n(R)) = M_n(J)$. Conclude that if I is an ideal contained in J then $A \in M_n(R)$ is invertible iff its image in $M_n(R/I)$ is invertible.

b) Prove that if R is commutative then $J(R[x]) = N[x]$, where N is the nilradical of R (problem 6 from third homework may be useful).

c) Prove that if R has no non-zero nil ideals (i.e. two sided ideals whose all elements are nilpotent) then $J(R[x]) = \{0\}$. Hint: Consider a non-zero polynomial of lowest degree in $J(R[x])$ and show that its leading coefficient commutes with all the other coefficients. Use this to show that the ideal in R generated by the leading leading coefficient is nil.

d) For a left R module M define $\text{rad}(M)$ to be the intersection of all maximal submodules of M (set $\text{rad}(M) = M$ if M has no maximal submodules). Prove that $JM \subseteq \text{rad}(M)$ for any left R-module M. Prove that the equality holds for projective modules M. Hint: Show that $\text{rad}(M \oplus N) = \text{rad}(M) \oplus \text{rad}(N)$, and, more generally, $\text{rad}(\bigoplus_{i\in I} M_i) = \bigoplus_{i\in I} \text{rad}(M_i)$.

Problem 5. Let R be a commutative domain (integral domain).

a) Let I be an ideal of R which is invertible. Prove that if M is any divisible R-module and $f : I \longrightarrow M$ a homomorphism of R-modules then f can be extended to a homomorphism $f : R \longrightarrow M$.

b) Let R be a Dedekind domain. Prove that any divisible R-module is injective.

Remark. The converse is also true: if R is a commutative domain such that every divisible R-module is injective then R is a Dedekind domain.

Problem 6. Let $R = \mathbb{Z}[x]$. Let $M = \mathbb{Q}(x)/\mathbb{Z}[x]$, where $\mathbb{Q}(x)$ is the field of rational functions with rational coefficients. Prove that M is a divisible $\mathbb{Z}[x]$-module. Consider the ideal $I = (2, x)$ of $\mathbb{Z}[x]$. Prove that there is a $\mathbb{Z}[x]$-module homomorphism $f : I \longrightarrow M$ such that $f(2) = \frac{1}{x} + \mathbb{Z}[x]$ and $f(x) = \frac{x}{2} + \mathbb{Z}[x]$. Use it to prove that M is not injective $\mathbb{Z}[x]$-module.