The Norm and the Trace

Let L/K be a finite extension of fields of degree $d = [L : K]$. Consider a finite dimensional vector space V over L of dimension n and a liner transformation $T : V \longrightarrow V$.

We may consider V as a vector space over K. If $v_1, \ldots, v_n$ is a basis of V as a vector space over L and $a_1, \ldots, a_d$ is a basis of L over K then it is easy to see that $a_i v_j$, $1 \le i \le d$, $1 \le j \le n$ is a basis of V as a vector space over K. Thus the dimension of V over K is nd. The L-linear transformation T can be considered as a K-linear transformation. Let $\det_L T$, $\mathrm{tr}_L T$ be the determinant and trace of T considered as a L-liner transformation and let $\det_K T$, $\mathrm{tr}_K T$ be the determinant and trace of T considered as a K-liner transformation. Our goal is to understand how these numbers are related.

Fix a basis $v_1, \ldots, v_n$ of V and a basis $a_1, \ldots, a_d$ of L over K. Then $a_1v_1, a_2v_1, \ldots, a_dv_1, a_1v_2, \ldots, a_dv_n$ is a basis of V over K.

Recall that an elementary linear transformation (in the chosen basis $v_1, \ldots, v_n$ of the L-vector space V) is one of the following linear transformations:

- for $i \neq j$ and $a \in L$ define ${}_aT_{i,j}$ as the linear transformation which maps v_j to $av_i + v_j$ and maps v_k to v_k for all $k \neq j$ (so in the chosen basis the matrix of ${}_aT_{i,j}$ is the elementary matrix $E_{i,j}(a)$ with 1 on the diagonal, a in the (i, j)-entry and 0 everywhere else).
- for $0 \neq a \in L$ define ${}_aT_j$ as the linear transformation which maps v_j to av_j and maps v_k to v_k for all $k \neq j$ (so in the chosen basis the matrix of ${}_aT_j$ is the elementary matrix $S_j(a)$ with a in the (j, j)-entry, 1 in all other diagonal entries and 0 everywhere else).

We have $\det_L {}_aT_{i,j} = 1$ and $\det_L {}_aT_i = a$. In order to compute the determinants over K we find the matrices of elementary transformations in the K-basis $a_1v_1, a_2v_1, \ldots, a_dv_1, a_1v_2, \ldots, a_dv_n$ of V. If $k \neq j$ then ${}_aT_{i,j}(a_lv_k) = a_l\ {}_aT_{i,j}(v_k) = a_lv_k$ and ${}_aT_j(a_lv_k) = a_l\ {}_aT_j(v_k) = a_lv_k$. Now

$$ {}_aT_{i,j}(a_lv_j) = a_l\ {}_aT_{i,j}(v_j) = a_l(av_i + v_j) = a_lv_j + aa_lv_i = a_lv_j + \sum_{s=1}^{d} k_{s,l}a_sv_i, $$

where $aa_l = \sum_{s=1}^{d} k_{s,l}a_s$ with $k_{s,l} \in K$. It follows that the matrix of ${}_aT_{i,j}$ in the basis $a_1v_1, a_2v_1, \ldots, a_dv_1, a_1v_2, \ldots, a_dv_n$ is upper-triangular if $i < j$ (lower-triangular if $i > j$) and has 1 on the diagonal. Thus $\det_K {}_aT_{i,j} = 1$. Similarly,

$$ {}_aT_j(a_lv_j) = a_l\ {}_aT_j(v_j) = a_l(av_j) = (aa_l)v_j = \sum_{s=1}^{d} k_{s,l}a_sv_i, $$

where $aa_l = \sum_{s=1}^{d} k_{s,l}a_s$ with $k_{s,l} \in K$. It follows that the matrix of ${}_aT_j$ in the basis $a_1v_1, a_2v_1, \ldots, a_dv_1, a_1v_2, \ldots, a_dv_n$ is block-diagonal, with n blocks of size $d \times d$, where the $j-$th block is the matrix $(k_{s,l})$ and the other blocks are identity matrices. Thus $\det_K {}_aT_j = \det(k_{s,l}) = N_{L/K}(a)$. Note that we can summarize the above computations as follows:

if T is an elementary transformation then $\det_K T = N_{L/K}(\det_L T)$

(since $N_{L/K}(1) = 1$).

Recall now that every invertible linear transformation is a product of elementary transformations (which is the same as saying that every invertible matrix is a product of elementary matrices). In other words, if T is invertible then we may write $T = T_1 \ldots T_m$, where each T_i is elementary. Thus

$$ \det_K T = \det_K T_1 \ldots \det_K T_m = N_{L/K}(\det_L T_1) \ldots N_{L/K}(\det_L T_m) = N_{L/K}(\det_L T_1 \ldots \det_L T_m) = N_{L/K}(\det_L T). $$

If T is not invertible then $\det_L T = 0 = \det_K T$. Thus we proved the following

Theorem 1. *For any L-linear transformation T we have* $\det_K T = N_{L/K}(\det_L T)$.

In order to prove an analogous result for the trace consider the transformations ${}_aT_{i,j} - I$ and ${}_{a+1}T_i - I$. Let us call such transformations *basic*. Note that every liner transformation is a sum of basic linar transfromations. From our discussion above we easily see that $\mathrm{tr}_L({}_aT_{i,j} - I) = 0 = \mathrm{tr}_K({}_aT_{i,j} - I)$, $\mathrm{tr}_L({}_{a+1}T_i - I) = a$, and $\mathrm{tr}_K({}_{a+1}T_i - I) = T_{L/K}(a)$. Thus $\mathrm{tr}_K(T) = T_{L/K}(\mathrm{tr}_L(T))$ for any basic T. Now if T is a sum $T_1 + \ldots + T_m$ of basic transfromations then

$$\mathrm{tr}_K(T_1 + \ldots + T_m) = \mathrm{tr}_K(T_1) + \ldots + \mathrm{tr}_K(T_m) = T_{L/K}(\mathrm{tr}_L(T_1)) + \ldots + T_{L/K}(\mathrm{tr}_L(T_m)) =$$

$$= T_{L/K}(\mathrm{tr}_L(T_1) + \ldots + \mathrm{tr}_L(T_m)) = T_{L/K}(\mathrm{tr}_L(T)).$$

Thus we have proved the following

Theorem 2. *For any L-linear transformation T we have* $\mathrm{tr}_K T = T_{L/K}(\mathrm{tr}_L T)$.

Suppose now that $K \subseteq L \subseteq M$ are fields and M/K is finite. Let $a \in M$. We may consider the special case of Theorem 1 when $V = M$ and T is the multiplication by a. Then we get $\det_K T = N_{M/K}(a)$, $\det_L T = N_{M/L}(a)$, $\mathrm{tr}_K T = T_{M/K}(a)$, $\mathrm{tr}_L T = T_{M/L}(a)$. Thus we get

Theorem 3. *Let $K \subseteq L \subseteq M$ be fields such that M/K is finite. For any $a \in M$ we have*

$$N_{M/K}(a) = N_{L/K}(N_{M/L}(a))$$

and

$$T_{M/K}(a) = T_{L/K}(T_{M/L}(a)).$$