Problem set 1

Problem 1. Let $\mathcal{J}$ be a small category and suppose that limits (colimits) of shape $\mathcal{J}$ exist in $\mathcal{C}$. Let $\mathcal{K}$ be another small category and H a functor from $\mathcal{J}$ to $\mathcal{F}un(\mathcal{K}, \mathcal{C})$. For any object a of $\mathcal{K}$ we have the functor $E_a : \mathcal{F}un(\mathcal{K}, \mathcal{C}) \longrightarrow \mathcal{C}$ which sends any functor G to $G(a)$ (evaluation at a).

Prove that the limit (colimit) of H exists and $(\varprojlim H)(a) = \varprojlim(E_a \circ H)$ $((\varinjlim H)(a) = \varinjlim(E_a \circ H))$. This means that limits and colimits in functor categories can be computed pointwise. This result allows the following interpretation of the result that limits commute with limits and colimits commute with colimits. First, note that we have the natural identifications:

$$\mathcal{F}un(\mathcal{J}, \mathcal{F}un(\mathcal{K}, \mathcal{C})) \approx \mathcal{F}un(\mathcal{J} \times \mathcal{K}, \mathcal{C}) \approx \mathcal{F}un(\mathcal{K}, \mathcal{F}un(\mathcal{J}, \mathcal{C})).$$

Suppose that limits (colimits) of shape $\mathcal{J}$ and shape $\mathcal{K}$ exist in $\mathcal{C}$. Let $F : \mathcal{J} \times \mathcal{K} \longrightarrow \mathcal{C}$ be a functor. Then

$$\varprojlim_{\mathcal{J}}(\varprojlim_{\mathcal{K}} F(j,k)) = \varprojlim_{\mathcal{K}}(\varprojlim_{\mathcal{J}} F(j,k)) = \varprojlim_{\mathcal{J} \times \mathcal{K}} F;$$

$$\varinjlim_{\mathcal{J}}(\varinjlim_{\mathcal{K}} F(j,k)) = \varinjlim_{\mathcal{K}}(\varinjlim_{\mathcal{J}} F(j,k)) = \varinjlim_{\mathcal{J} \times \mathcal{K}} F.$$

Problem 2. Let $\mathcal{J}$ be a small category. We say that $\mathcal{J}$ is *filtered* if it satisfies the following two properties:

1. For any two objects a, b in $\mathcal{J}$ there is an object c in $\mathcal{J}$ and morphisms $f : a \longrightarrow c$, $g : b \longrightarrow c$.
2. for any two morphisms $f, g : a \longrightarrow b$ in $\mathcal{J}$ there is a morphisms $h : b \longrightarrow c$ such that $hf = hg$.

We say that a functor $F : \mathcal{J} \longrightarrow \mathcal{C}$ is *filtered* if $\mathcal{J}$ is filtered. The goal of this exercise is to study colimits of filtered functors into $\mathcal{S}et$ (the category of sets).

a) Let $F : \mathcal{J} \longrightarrow \mathcal{S}et$ be a filtered functor. Suppose that X is a set and we have functions $f_a : F(a) \longrightarrow X$ for each object a of $\mathcal{J}$ such that $f_b F(f) = f_a$ for any morphism $f : a \longrightarrow b$ in $\mathcal{J}$. Prove that the following two properties are equivalent:

(i) For any object a in $\mathcal{J}$ and any $u, w \in F(a)$ such that $f_a(u) = f_a(w)$ there is a morphism $h : a \longrightarrow b$ in $\mathcal{J}$ such that $F(h)(u) = F(h)(w)$.

(ii) For any objects a, b in $\mathcal{J}$ and any $u \in F(a)$, $w \in F(b)$ such that $f_a(u) = f_b(w)$ there are morphisms $h_1 : a \longrightarrow c$ and $h_2 : b \longrightarrow c$ in $\mathcal{J}$ such that $F(h_1)(u) = F(h_2)(w)$.

b) Suppose X and the functions $f_a : F(a) \longrightarrow X$ for each object a of $\mathcal{J}$ satisfy the equivalent conditions (i), (ii) in a) and also the condition

(iii) For every $x \in X$ there is an object a of $\mathcal{J}$ and $u \in F(a)$ such that $x = f_a(u)$.

Prove that X with the functions f_a is a colimit of F.

c) Let $F : \mathcal{J} \longrightarrow \mathcal{S}et$ be a filtered functor. Consider the disjoint union Z of the sets $F(a)$, a an object in $\mathcal{J}$. Define the following relation $\sim$ on Z: $u \sim w$ if $u \in F(a)$, $w \in F(b)$ and there are morphisms $h_1 : a \longrightarrow c$ and $h_2 : b \longrightarrow c$ in $\mathcal{J}$ such that $F(h_1)(u) = F(h_2)(w)$. Prove that $\sim$ is an equivalence relation and the quotient set $Z/\sim$ with the functions $f_a : F(a) \hookrightarrow Z \longrightarrow Z/\sim$ is a colimit of F.

d) Let $\mathcal{J}$ be a small filtered category and $\mathcal{K}$ a finite category (morphisms form a finite set). Let H be a functor from $\mathcal{J} \times \mathcal{K}$ to $\mathcal{S}et$. Prove that

$$\varinjlim_{\mathcal{J}} (\varprojlim_{\mathcal{K}} H(j,k)) = \varprojlim_{\mathcal{K}} (\varinjlim_{\mathcal{J}} F(j,k)).$$

In light of the discussion in Problem 1, this means that in $\mathcal{S}et$ filtered colimits commute with finite limits. Show that in general, limits and colimits do not commute in $\mathcal{S}et$.

e) Let $\mathcal{J}$ be a small filtered category and let F be a functor from $\mathcal{J}$ to the category $\mathcal{G}r$ of groups. Composing with the forgetful functor, we get a functor F' from $\mathcal{J}$ to $\mathcal{S}et$. Let X be a colimit of F'. Show that X has a natural group structure which makes it a colimit of F.

Problem 3. Let X be a set with two binary operations $\square$ and $\triangle$, each having two-sided identity element. Suppose that for any x, y, u, w in X we have

$$(x\square y)\triangle(u\square w) = (x\triangle u)\square(y\triangle w).$$

Prove that the two operations coincide, are commutative and associative (this is often called the Eckmann-Hilton argument).

Before the next problem, let us make two useful remarks about products.

a)Let $f : X \longrightarrow A$ and $g : X \longrightarrow B$ be two morphisms and suppose the product $A \times B$ exists. Then there is unique morphism $f \times g : X \longrightarrow A \times B$ such that $\pi_A \circ (f \times g) = f$ and $\pi_B \circ (f \times g) = g$. This is just a definition of a product. In particular, when $X = A = B$ a $f = g = id_A$ then the corresponding morphisms $id_A \times id_A$ is denoted by Δ and called *the diagonal*.

b)Let $f : X \longrightarrow A$ and $g : Y \longrightarrow B$ be two morphisms and suppose the products $X \times Y$ and $A \times B$ exists. There is unique morphisms $(f, g) : X \times Y \longrightarrow A \times B$ such that $\pi_A \circ (f, g) = f \circ \pi_X$ and $\pi_B \circ (f, g) = g \circ \pi_Y$.

Exercise. Make analogous observations for the coproduct.

Problem 4. Consider a category $\mathcal{C}$. Let $\Phi : \mathcal{G}r \longrightarrow \mathcal{S}et$ be the forgetful functor from groups to sets. A *group structure* on an object A of $\mathcal{C}$ is a functor $F : \mathcal{C}^o \longrightarrow \mathcal{G}r$ such that $\Phi \circ F$ is isomorphic to the functor $h_A = Hom_{\mathcal{C}}(-, A)$. In other words, it is a functorial way to equip all sets $Hom_{\mathcal{C}}(X, A)$ with a group structure. We say that A is a group object in $\mathcal{C}$ (but note that often there are many group structures on A so it is not enough to just specify the object A).
Suppose we have a group structure on an object A of $\mathcal{C}$.

a) Prove that there is unique morphism $i : A \longrightarrow A$ such that for every X, the "composition with i" map $Hom_{\mathcal{C}}(X, A) \longrightarrow Hom_{\mathcal{C}}(X, A)$ coincides with the inversion map in the group $Hom_{\mathcal{C}}(X, A)$ (use Yoneda's Lemma).

b) Suppose that the product $A \times A$ exists in $\mathcal{C}$. Thus we have natural isomorphisms

$$Hom_{\mathcal{C}}(X, A \times A) \approx Hom_{\mathcal{C}}(X, A) \times Hom_{\mathcal{C}}(X, A).$$

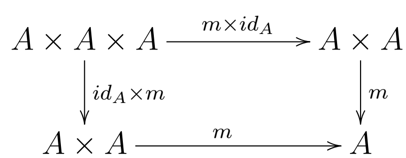
Prove that there is a unique morphism $m : A \times A \longrightarrow A$ such that for every X the group structure on $Hom_{\mathcal{C}}(X, A)$ is given by:

$$Hom_{\mathcal{C}}(X, A) \times Hom_{\mathcal{C}}(X, A) \approx Hom_{\mathcal{C}}(X, A \times A) \longrightarrow Hom_{\mathcal{C}}(X, A),$$

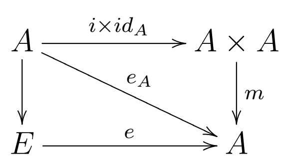
where the last arrow is the "composition with m" map.

c) Suppose that $\mathcal{C}$ has a final object E. Prove that there is unique morphisms $e : E \longrightarrow A$ such that for every X the composition e_X of the unique morphism $X \longrightarrow E$ with e is the identity of the group $Hom_{\mathcal{C}}(X, A)$.

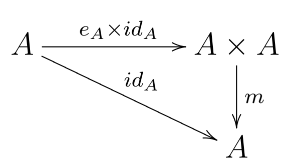
d) Suppose now that the products $A \times A$ and $A \times A \times A$ exist and E is a final object in $\mathcal{C}$. We use the notation established in parts a), b), c). Prove that the associativity of multiplication implies that $m \circ (m \times id_A) = m \circ (id_A \times m)$, i.e. the following diagram commutes:



Show that the left inverse property in groups implies that $m \circ (i \times id_A) = e_A$, i.e. the following diagram commutes:



Show that the left identity property in groups implies that $m \circ (e_A \times id_A) = id_A$, i.e. the following diagram commutes:



Conversely, suppose that A is an object in a category $\mathcal{C}$ such that $A \times A$ and $A \times A \times A$ exist and E is a final object. Let $m : A \times A \longrightarrow A$ be a morphism for which there exists morphisms $e : E \longrightarrow A$ and $i : A \longrightarrow A$ such that the above three diagrams commute. Prove that e and i are unique and m defines a group structure on A via

$$Hom_{\mathcal{C}}(X, A) \times Hom_{\mathcal{C}}(X, A) \approx Hom_{\mathcal{C}}(X, A \times A) \longrightarrow Hom_{\mathcal{C}}(X, A),$$

where the last arrow is the "composition with m" map.

e) Let $\mathcal{C}$ be a category with finite products (this includes the final object as the empty product). Part d) tells us that group structures on an object A are in bijection with triples m, i, e for which the three diagrams in part d) commute. We can consider the category $\mathcal{GC}$ of group objects in $\mathcal{C}$ whose objects are quadruples (A, m, i, e) for which the three diagrams in d) commute and a morphism $(A, m, i, e) \longrightarrow (B, m', i', e')$ is just morphisms $f : A \longrightarrow B$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{array}{ccc} A \times A & \xrightarrow{m} & A \\ \downarrow{\scriptstyle (f,f)} & & \downarrow{\scriptstyle f} \\ B \times B & \xrightarrow{m'} & B \end{array}$$

Show that the commutativity of this diagram is equivalent to the condition that for every X the "composition with f" map

$$Hom_{\mathcal{C}}(X, A) \longrightarrow Hom_{\mathcal{C}}(X, B)$$

is a group homomorphism. Conclude that we have $fi = i'f$ and $fe = e'$. Show that $\mathcal{D}$ is another category with finite products and $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor which preserves finite products then F induces a functor $F : \mathcal{GC} \longrightarrow \mathcal{GD}$. Suppose $G : \mathcal{D} \longrightarrow \mathcal{C}$ is right adjoint to F (so G automatically preserves products). Show that $G : \mathcal{GD} \longrightarrow \mathcal{GC}$ is right adjoint to $F : \mathcal{GC} \longrightarrow \mathcal{GD}$

f) Consider the category $\mathcal{G}r$ of groups. Prove that if (A, m, i, e) is a group object in $\mathcal{G}r$ then A is an abelian group and m is the multiplication of A (the Eckmann-Hilton argument should be helpful).

f) A *cogroup structure* on an object A of $\mathcal{C}$ is the same as a group structure on A in the dual category $\mathcal{C}^o$. This is the same as a functorial way to equip all sets $Hom_{\mathcal{C}}(A, X)$ with a group structure. State the conditions analogous to what was done in part d) for a cogroup structure on A (in terms of three diagrams in $\mathcal{C}$, assuming $\mathcal{C}$ has finite coproducts).

g) Find all cogroups in the category $\mathcal{S}et$.

h) Suppose A has a cogroup structure and B has a group structure in a category $\mathcal{C}$. It follows that the set $Hom_{\mathcal{C}}(A, B)$ has two group structures. Prove that these two structures coincide and are abelian (the Eckmann-Hilton argument should be helpful). Can you explain g) using this observation? (**Remark.** This in particular explains why higher homotopy groups are abelian).