Problem set 2

Problem 1. Let $\mathcal{A}$ be an abelian category. Recall that and object M in $\mathcal{A}$ is called **projective** if $Hom(M,-)$ is an exact functor from $\mathcal{A}$ to the category $\mathcal{A}b$ of abelian groups. M is called a **generator** if the functor $Hom(M,-)$ is faithfull. We assume that $\mathcal{A}$ has all coproducts and a projective generator.

a) Show that the coproduct of a collection of objects is projective if and only if all the objects in the coproduct are projective. Show that coproduct of generators is a generator. State and prove the dual to these results.

b) Let M be any object of $\mathcal{A}$. Let $R = Hom(M,M)$. Then R is a (unital, associative) ring. Show that each abalian group $Hom(M,A)$ carries a natural right $R-$module structure, where $fr = f \circ r$ (composition) for $f \in Hom(M,A)$ and $r \in Hom(M,M)$. Thus the functor $Hom(M,-)$ can be considered as a functor from $\mathcal{A}$ to $mod-R$ (the categery of right R-modules).

c) Suppose that Q is a generator. For any object A consider the coproduct

$$Q_A = \bigoplus_{f \in Hom(Q,A)} Q_f,$$

where each Q_f is Q. The collection of morphisms $f : Q_f = Q \longrightarrow A$ induces a morphism $f_A : Q_A \longrightarrow A$. Prove that f_A is an epimorphism. Suppose now that S is a set of objects in $\mathcal{A}$. Consider the coproduct

$$Q_S = \bigoplus_{A \in S} Q_A.$$

For $A \in S$ we have a morphism $f_A : Q_A \longrightarrow A$ and for $B \neq A$ we have the zero morphisms $Q_B \longrightarrow A$. This collection of morphisms corresponds to a morphism $Q_S \longrightarrow A$. Show that this morphism is an epimorphism.

d) Let $\mathcal{B}$ be a small, full, exact subcategory of $\mathcal{A}$. Parts a) and c) show that there is a projective generator P which has an epimorphism to every object of $\mathcal{B}$. The functor $T(-) = Hom(P,-)$ is a faithfull exact functor from $\mathcal{B}$ to $mod-R$, where $R = T(P)$ (see part b)). Let A, B be objects in $\mathcal{B}$. Consider a right $R-$module homomorhism $f : T(A) \longrightarrow T(B)$. There are epimorphisms $\pi : P \longrightarrow A$ and $\sigma : P \longrightarrow B$. Let

$\eta : K \longrightarrow P$ be the kernel of π, so $0 \longrightarrow K \xrightarrow{\eta} P \xrightarrow{\pi} A \longrightarrow 0$ is exact. Since T is exact, we have the following diagram of R-modules with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T(K) & \xrightarrow{T(\eta)} & T(P) = R & \xrightarrow{T(\pi)} & T(A) & \longrightarrow & 0 \\ & & & & & & \downarrow f & & \\ & & & & T(P) = R & \xrightarrow{T(\sigma)} & T(B) & \longrightarrow & 0 \end{array}$$

Prove that there is an R-module homomorphisms $h : R \longrightarrow R$ such that $T(\sigma)h = fT(\pi)$. Show that $h = T(u)$ for some $u : P \longrightarrow P$. Then show that $\sigma u \eta = 0$ and conclude that there is $\phi : A \longrightarrow B$ such that $\phi\pi = \sigma u$. Show that $T(\phi) = f$. This proves that the functor T is full.

Problem 2. a) Explain why pre-additive categories with one object are the same as rings.

b) Let $\mathcal{M}$ be a small pre-additive category and $\mathcal{A}$ an abelian category. Prove that additive functors from $\mathcal{M}$ to $\mathcal{A}$ form an abelian category. What is this category when $\mathcal{M}$ has one object and $\mathcal{A} = \mathcal{A}b$ is the category of abelian groups?

c) Let $\mathcal{C}$ be a small category and $\mathcal{A}$ an abelian category. Prove that all functors from $\mathcal{C}$ to $\mathcal{A}$ form an abelian category.

Problem 3. Let $\mathcal{A}$ be an abelian category. Consider the category $SES(\mathcal{A})$ of short exact sequences in $\mathcal{A}$. Prove that this is a pre-abelian category, but it is not abelian if $\mathcal{A}$ has a non-zero object.

Problem 4. Let $\mathcal{A}$ be an abelian category. Recall that for any object X of $\mathcal{A}$ we defined the collections of sub-objects $S(X)$ and quotient-objects $Q(X)$ of X. Both $S(X)$ and $Q(X)$ have a natural order and there is order reversing bijection from $S(X)$ to $Q(X)$. Any two sub-objects of X have the largest lower bound, called the intersection of the sub-objects and they have the least upper bound called the sum of the sub-objects (there are dual statements for the quotient objects). The problem is that in general we do not know if $S(X)$ is a set (and it does not have to be a set).

a) Suppose that $\mathcal{A}$ has a generator G. For any monomorphism $i : A \hookrightarrow X$ let $M(i)$ be the image of the map $Hom(G, A) \longrightarrow Hom(G, X)$ induced by i. Prove that $i \leq j$ if and only if $M(i) \subseteq M(j)$ (recall that $i \leq j$, where $i : A \hookrightarrow X$, $j : A \hookrightarrow X$

are monomorphisms, means that there is $k : A \longrightarrow B$ such that $i = jk$). Conclude that i and j are equivalent if and only if $M(i) = M(j)$. Conclude that both $S(X)$ and $Q(X)$ are sets.

From now on we assume that $\mathcal{A}$ has a generator and all coproducts (hence also all colimits).

b) For any set of sub-objects $i_\alpha : A_\alpha \hookrightarrow X$ we have a corresponding morphism $\bigoplus_\alpha A_\alpha \longrightarrow X$. The image of this morphism is a sub-object of X denoted by $\sum_\alpha A_\alpha$. Prove that $\sum_\alpha A_\alpha$ is the lest upper bound of the sub-object A_α in $S(X)$.

c) Suppose that we have a linearly ordered set of sub-object $i_\alpha : A_\alpha \hookrightarrow X$ of X and let $A = \sum S_\alpha$ (which should be called the union in this case), so we have monomorphisms $k_\alpha : A_\alpha \hookrightarrow A$. Suppose that we have morphisms $f_\alpha : A_\alpha \longrightarrow B$ such that if $i_\alpha \leq i_\beta$ then $f_\beta k_{\alpha,\beta} = f_\alpha$, where $k_{\alpha,\beta} : A_\alpha \hookrightarrow A_\beta$ is such that $i_\alpha = i_\beta k_{\alpha,\beta}$. We would like to have a morphisms $f : A \longrightarrow B$ such that $fk_\alpha = f_\alpha$ (this is clearly true in any category of modules). Prove that this holds for any B if and only if A is the colimit of the diagram formed by A_α and the monomorphisms $k_{\alpha,\beta}$.

Before we state the next part let us recall some facts about colimits and limits. Let $\mathcal{C}$ be a small category. Suppose all colimits (limits) of shape $\mathcal{C}$ exist in $\mathcal{A}$. The category $\mathcal{F}un(\mathcal{C}, \mathcal{A})$ of all functors from $\mathcal{C}$ to $\mathcal{A}$ is an abelian category and the colimit (limit) is a functor from $\mathcal{F}un(\mathcal{C}, \mathcal{A})$ to $\mathcal{A}$. The colimit (limit) is left adjoint (right adjoint) to the diagonal functor from $\mathcal{A}$ to $\mathcal{F}un(\mathcal{C}, \mathcal{A})$. It follows that the colimit (limit) is right exact (left exact).

d) Prove that the following conditions are equivalent:

1. any A as in part c) is the colimit of the diagramm given by A_α
2. for any X, any sub-object K of X, and any linearly ordered sub-objects A_α of X we have $(\sum_\alpha A_\alpha) \cap K = \sum (A_\alpha \cap K)$.
3. for any X, any sub-object K of X, and any lattice of sub-objects A_α of X we have $(\sum_\alpha A_\alpha) \cap K = \sum (A_\alpha \cap K)$.
4. all filtered colimits in $\mathcal{A}$ are exact functors.

Abelian categories which have a generator and all coproducts, and in which all filtered colimits are exact are called **Grothendieck abelian categories**.

e) Let $\mathcal{A}$ be a Grothendieck abelian category. Prove that a contravariant functor F from $\mathcal{A}$ to $\mathcal{S}et$ is representable if and only if it commutes with colimits i.e. $F(\varinjlim G) = \varprojlim(F \circ G)$. Conculde that all products exist in $\mathcal{A}$. Thus all limits exist in $\mathcal{A}$

f) Show that if $\mathcal{A}$ and $\mathcal{A}^o$ are both Grothendieck abelian catgeories then all objects in $\mathcal{A}$ are zero.

g) Show that any coproduct is exact in a Grothendieck abelian category. Hint: Finite coproducts are also products, hence they are exact. Show that a coproduct $\bigoplus_{i \in T} X_i$ can be considered as a filtered colimit of the finite coproducts $\bigoplus_{i \in S} X_i$ where S ranges over all finite subsets of T. Show that products do not need to be exact.

Problem 5. Let $\mathcal{A}$ be an abelian category. Recall that a simplicial object in $\mathcal{A}$ is a functor from Δ^o to $\mathcal{A}$. Thus the simplicial objects in $\mathcal{A}$ form an abelian category denoted by $\Delta^o\mathcal{A}$. More explicitly, a simplicial object $C_\bullet$ in $\mathcal{A}$ is a collection of objects C_n for $n = 0, 1, \ldots$ and morphisms $C(f) : C_n \longrightarrow C_m$ for every non-decreasing function $f : \{0, 1, \ldots, m\} \longrightarrow \{0, 1, \ldots, n\}$ such that $C(fg) = C(g)C(f)$ whenever the composition is defined. In particular, we have the face morphisms $\delta_i : C_n \longrightarrow C_{n-1}$ for $i = 0, 1, \ldots n$ and degeneracy morphisms $\sigma_i : C_{n-1} \longrightarrow C_n$ for $i = 0, 1, \ldots n-1$. Define $A^n = C_{-n}$ for $n \leq 0$ and $A^n = 0$ for $n > 0$. Then set $d^n = \sum_{i=0}^n \partial_i : A^n \longrightarrow A^{n+1}$ for $n < 0$ and $d^n = 0$ for $n \geq 0$.

a) Prove that $A^\bullet : \ldots \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \xrightarrow{d^{n+1}} A^{n+2} \xrightarrow{d^{n+2}} \ldots$ is a complex. The assignment $K : C_\bullet \mapsto A^\bullet = K(C_\bullet)$ is a functor $K : \Delta^o\mathcal{A} \longrightarrow \mathcal{K}om^-(\mathcal{A})$ (how is it defined on morphisms?).

b) Let B^n be the sub-object of A^n which is the intersection of the kernels of the morphisms $\partial_i : A^n \longrightarrow A^{n+1}$, $i = 0, 1, \ldots n-1$ (when $n < 0$; for $n \geq 0$ set $B^n = 0$). Show that d^n induces a morphism $B^n \longrightarrow B^{n+1}$ which coincides with $(-1)^n \partial_n$ (restricted to B^n). Thus $B^\bullet$ is a subcomplex of $A^\bullet$ denoted by $N(C_\bullet)$. Show that this defines a functor $N : \Delta^o\mathcal{A} \longrightarrow \mathcal{K}om^-(\mathcal{A})$.

c) Let D^n be the sub-object of A^n which is the lest upper bound (sum) of the images of the degeneracy maps $\sigma_i : A_{n+1} \longrightarrow A_n$, $i = 0, 1, \ldots, n-1$ ($D^n = 0$ for $n \geq 0$). Show that d^n induces a morphism $D^n \longrightarrow D^{n+1}$. Thus $D^\bullet$ is a subcomplex of $A^\bullet$ denoted by $D(C_\bullet)$. Show that this defines a functor $D : \Delta^o\mathcal{A} \longrightarrow \mathcal{K}om^-(\mathcal{A})$.

d) Prove that A^n is the direct sum of B^n and D^n. Thus $K(C_\bullet) = N(C_\bullet) \oplus D(C_\bullet)$.

e) Prove that the complex $D(C_\bullet)$ is acyclic.

f) Let $X_\bullet$ be a simplicial set. Let A_n be the free abelian group on the set X_n. Each function $X(f) : X_n \longrightarrow X_m$ extends to a group homomorphism $A_n \longrightarrow A_m$. Show that this defines a simplicial abelian group which is denoted by $\mathbb{Z}[X_\bullet]$. This way we get a functor $\Delta^o\mathcal{S}et \longrightarrow \Delta^o\mathcal{A}b$. The n-th cohomology of the complex $K(\mathbb{Z}[X_\bullet])$ is denoted by $H_{-n}(X_\bullet, \mathbb{Z})$ and called the $(-n)$-th homology of $X_\bullet$ with integer coefficients. When applied to the singular simplicial set of a topological space T we get the homology $H_n(T, \mathbb{Z})$. When applied to the classifying space BG of a group G (this is a simplicial set) we get the homology $H_n(G, \mathbb{Z})$.