1. δ-FUNCTORS

Let $\mathcal{C}$, $\mathcal{D}$ be abelian categories. Recall that for any abelian category $\mathcal{A}$ we have the category $SES(\mathcal{A})$ of short exact sequences in $\mathcal{A}$ and three functors $T_i : SES(\mathcal{A}) \longrightarrow \mathcal{A}$, $i = 1, 2, 3$, defined by

$$T_i(0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 0) = A_i.$$

Following Grothendieck, we make the following definitions

Definition 1. *A* **δ-functor** *form $\mathcal{C}$ to $\mathcal{D}$ is a collection of additive functors $F^n : \mathcal{C} \longrightarrow \mathcal{D}$, $n \in \mathbb{Z}$, and natural transformations $\delta^n : F^n \circ T_3 \longrightarrow F^{n+1} \circ T_1$ of functors from $SES(\mathcal{C})$ to $\mathcal{D}$ such that for every short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in $\mathcal{C}$ the sequence*

$$... \longrightarrow F^{n-1}(C) \xrightarrow{\delta^{n-1}} F^n(A) \longrightarrow F^n(B) \longrightarrow F^n(C) \xrightarrow{\delta^n} F^{n+1}(A) \longrightarrow ...$$

is a complex. If this complex is exact for every short exact sequence in $\mathcal{C}$ then the δ-functor is called **exact**.

Definition 2. *A δ-functor is called* **cohomological** *(***homological***) if $F^n = 0$ for all $n < 0$ (resp. for all $n > 0$).*

Remark. For a homological δ-functor it is customary to write F_n for F^{-n}, etc.

Definition 3. *A cohomological δ-functor is called* **effaceable** *if for any object X of $\mathcal{C}$ there is a monomorphism $u : X \longrightarrow Y$ such that $F^n(u) = 0$ for all $n > 0$. A homological δ-functor is called* **effaceable** *if for any object X of $\mathcal{C}$ there is an epimorphism $u : Y \longrightarrow X$ such that $F^n(u) = 0$ for all $n < 0$. Any such u is called an* **effacing** *morphism.*

Definition 4. *A cohomological (homological) δ-functor is called* **universal**, *if it is exact and effaceable.*

Theorem 1. *Let (F^n) be a universal cohomological (homological) δ-functor and (G^n) any δ-functor. Any natural transfromation $\alpha : F^0 \longrightarrow G^0$ of functors (resp. $\alpha : G^0 \longrightarrow F^0$) uniquely extends to a natural transformation of δ-functors.*

Remark. A natural transformation of δ-functors is a collection of natural transformations $\alpha^n : F^n \longrightarrow G^n$, $n \in \mathbb{Z}$, such that for every k and every short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in $\mathcal{C}$ the following diagram commutes:

$$\begin{tikzcd} F^{k-1}(C) \arrow[r, "\delta^{k-1}"] \arrow[d, "\alpha^{k-1}"] & F^k(A) \arrow[d, "\alpha^k"] \\ G^{k-1}(C) \arrow[r, "\delta^{k-1}"] & G^k(A) \end{tikzcd} \tag{1}$$

Proof: We give a proof for cohomological δ-functors and leave the case of homological δ-functors as an exercise.

Uniqueness. Suppose we have two natural transformations (α^n), (β^n) such that $\alpha^0 = \alpha = \beta^0$. Clearly $\alpha^n = 0 = \beta^n$ for all $n < 0$, since $F^n = 0$ for such n. We show that $\alpha^n = \beta^n$ by induction on n. For $n = 0$ it is our assumption. Suppose then that $\alpha^k = \beta^k$ for all $k < n$. Let X be an object of $\mathcal{C}$ and $u : X \longrightarrow Y$ an effacing monomorphism. Thus we have an exact sequence

$$0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{j} Z \longrightarrow 0$$

where $Z = \operatorname{coker} u$, which leads to the following commutative diagram

$$\begin{tikzcd} F^{n-1}(Z) \arrow[r, "\delta^{n-1}"] \arrow[d, "\alpha^{n-1}=\beta^{n-1}"'] & F^n(X) \arrow[r, "F^n(u)"] \arrow[d, shift right, "\alpha^n"'] \arrow[d, shift left, "\beta^n"] & F^n(Y) \\ G^{n-1}(Z) \arrow[r, "\delta^{n-1}"] & G^n(X) \arrow[r, "G^n(u)"] & G^n(Y) \end{tikzcd}$$

with the top row exact. Note that $\alpha^n \delta^{n-1} = \delta^{n-1} \alpha^{n-1} = \delta^{n-1} \beta^{n-1} = \beta^n \delta^{n-1}$. Since $F^n(u) = 0$, the morphism δ_{n-1} in the top row is an epimorphism, hence the equality $\alpha^n \delta^{n-1} = \beta^n \delta^{n-1}$ implies that $\alpha^n = \beta^n$.

Existence. Suppose that we have already constructed natural transformations α^k for $k < n$ such that the diagrams (1) commute for $k < n$. Let X be an object of $\mathcal{C}$ and $u : X \longrightarrow Y$ an effacing monomorphism. Thus we have an exact sequence

$$0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{j} Z \longrightarrow 0$$

where $Z = \operatorname{coker} u$, which leads to the following commutative diagram

$$\begin{tikzcd}
F^{n-1}(Y) \arrow[r, "F^{n-1}(j)"] \arrow[d, "\alpha^{n-1}"'] & F^{n-1}(Z) \arrow[r, "\delta^{n-1}"] \arrow[d, "\alpha^{n-1}"'] & F^n(X) \arrow[r] & 0 \\
G^{n-1}(Y) \arrow[r, "G^{n-1}(j)"] & G^{n-1}(Z) \arrow[r, "\delta^{n-1}"] & G^n(X) &
\end{tikzcd}$$

with the top row exact. Note that $(\delta^{n-1}\alpha^{n-1})F^{n-1}(j) = \alpha^{n-1}G^{n-1}(j)\delta^{n-1} = 0$ (since $G^{n-1}(j)\delta^{n-1} = 0$). It follows that there exists unique morphism α_u such that the diagram

$$\begin{tikzcd}
F^{n-1}(K) \arrow[r, "\delta^{n-1}"] \arrow[d, "\alpha^{n-1}"] & F^n(X) \arrow[d, "\alpha_u"] \\
G^{n-1}(K) \arrow[r, "\delta^{n-1}"] & G^n(X)
\end{tikzcd}$$

commutes. If the natural transformation α^n exists, then we must have $\alpha^n = \alpha_u$ by uniqueness. In particular, α_u should not depend on the effacing morphism u. We will show that this is indeed true and that setting $\alpha^n = \alpha_u$ indeed defines a natural transformation of functors for which the diagrams (1) commute for $k = n$.

Suppose that we are given a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "j"] \arrow[d] & Z \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & A \arrow[r, "w"] & B \arrow[r, "i"] & C \arrow[r] & 0
\end{tikzcd}$$

with exact rows and u, w effacing monomorphisms. This leads to the following cube

$$\begin{tikzcd}
F^{n-1}(Z) \arrow[rr, "\delta^{n-1}"] \arrow[dr, "\alpha^{n-1}"] \arrow[dd, "F^{n-1}(g)"'] & & F^n(X) \arrow[dr, "\alpha_u"] \arrow[dd, "F^n(f)"' near start] & \\
& G^{n-1}(Z) \arrow[rr, "\delta^{n-1}" near start] \arrow[dd, "G^{n-1}(g)"' near start] & & G^n(X) \arrow[dd, "G^n(f)"] \\
F^{n-1}(C) \arrow[rr, "\delta^{n-1}" near end] \arrow[dr, "\alpha^{n-1}"] & & F^n(A) \arrow[dr, "\alpha_w"] & \\
& G^{n-1}(C) \arrow[rr, "\delta^{n-1}"] & & G^n(A)
\end{tikzcd}$$

in which all faces except perhaps the right face commute and in which the top δ^{n-1} is an epimorphism. We claim that it follows that the right face commutes too, i.e. that $G^n(f)\alpha_u = \alpha_w F^n(f)$. In fact, since the top δ^{n-1} is an epimorphism, it is enough to show that $G^n(f)\alpha_u\delta^{n-1} = \alpha_w F^n(f)\delta^{n-1}$. The commutativity of the faces gives

$$G^n(f)\alpha_u\delta^{n-1} = G^n(f)\delta^{n-1}\alpha^{n-1} = \delta^{n-1}G^{n-1}(g)\alpha^{n-1} =$$

$$= \delta^{n-1}\alpha^{n-1}F^{n-1}(g) = \alpha_w\delta^{n-1}F^{n-1}(g) = \alpha_w F^n(f)\delta^{n-1},$$

which confirms our claim.

Suppose now that we have two short exact sequences

$$0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{j} Z \longrightarrow 0$$

and

$$0 \longrightarrow A \xrightarrow{w} B \xrightarrow{i} C \longrightarrow 0$$

with u, w effacing monomorphisms and a morphism $f : X \longrightarrow A$. We can not apply the previous paragraph directly since there is no reason in general for f to extend to a morphism of short exact sequences. Fortunately, there is a way around this problem. We construct a third exact sequence:

$$0 \longrightarrow X \xrightarrow{u\times wf} A \times Y \longrightarrow W \longrightarrow 0$$

(where W is a cokernel of $u \times wf$). The following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "u"] & Y \arrow[r, "j"] & Z \arrow[r] & 0 \\
0 \arrow[r] & X \arrow[r, "u\times wf"] \arrow[u, "id"] \arrow[d, "f"] & Y \times B \arrow[r] \arrow[u, "p_Y"] \arrow[d, "p_B"] & W \arrow[r] \arrow[u] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "w"] & B \arrow[r, "i"] & C \arrow[r] & 0
\end{tikzcd}$$

where p_B, p_Y are projections and the vertical arrows from W are uniquely determined by the universal property of a cokernel. Since F^n are additive, $F^n(u \times wf) = F^n(u) \times F^n(w)F^n(f) = 0 \times 0 = 0$ for $n > 0$. Thus $u \times wf$ is an effacing monomorphism. Applying the previous paragraph to the top two and bottom two exact sequences yields the equalities $G^n(id)\alpha_{u\times wf} = \alpha_u F^n(id)$ and $G^n(f)\alpha_{u\times wf} = \alpha_w F^n(f)$. Thus $\alpha_{u\times wf} = \alpha_u$ and $G^n(f)\alpha_u = \alpha_w F^n(f)$. In particular, we may take $A = X$ and $f = id$ and we see that $\alpha_u = \alpha_w$. This proves independence of the effacing monomorphism and allows us to define $\alpha^n = \alpha_u$. Since for any morphism f we have $G^n(f)\alpha^n = \alpha^n F^n(f)$, we see that α^n is a natural transformation from F^n to G^n.

It remains to show that the diagrams (1) commute for $k = n$. So we start with an exact sequence $0 \longrightarrow A \xrightarrow{i} B \longrightarrow C \longrightarrow 0$ in $\mathcal{C}$. Note that if i is an effacing monomorphism then the corresponding diagram (1) commutes for $k = n$ by definition of α^n. Let $u : B \longrightarrow Y$ be an effacing monomorphism. Note that $ui : A \longrightarrow Y$ is also an effacing monomorphism. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "id"] & B \arrow[r] \arrow[d, "u"] & C \arrow[r] \arrow[d, "w"] & 0 \\
0 \arrow[r] & A \arrow[r, "ui"] & Y \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

which leads to the following cube

$$\begin{tikzcd}
F^{n-1}(C) \arrow[rr, "\delta^{n-1}"] \arrow[dr, "\alpha^{n-1}"] \arrow[dd, "F^{n-1}(w)"'] & & F^n(A) \arrow[dr, "\alpha^n"] \arrow[dd, "F^n(id)" near start] & \\
& G^{n-1}(C) \arrow[rr, "\delta^{n-1}" near start] \arrow[dd, "G^{n-1}(w)"' near start] & & G^n(A) \arrow[dd, "G^n(id)"] \\
F^{n-1}(Z) \arrow[rr, "\delta^{n-1}" near start] \arrow[dr, "\alpha^{n-1}"] & & F^n(A) \arrow[dr, "\alpha^n"] & \\
& G^{n-1}(Z) \arrow[rr, "\delta^{n-1}"] & & G^n(A)
\end{tikzcd}$$

in which all faces except possibly the top one commute. Our proof will be completed if we show that the top face commutes as well. But this is quite simple at this point:

$$\alpha^n\delta^{n-1} = G^n(id)\alpha^n\delta^{n-1} = \alpha^n F^n(id)\delta^{n-1} = \alpha^n\delta^{n-1}F^{n-1}(w) =$$

$$= \delta^{n-1}\alpha^{n-1}F^{n-1}(w) = \delta^{n-1}G^{n-1}(w)\alpha^{n-1} = G^n(id)\delta^{n-1}\alpha^{n-1} = \delta^{n-1}\alpha^{n-1}. \quad \square$$

Corollary 1. *A universal cohomological (or homological) δ-functor (F^n) is uniquely determined by F^0. More precisely, if (G^n) is another universal cohomological (or homological) δ-functor and $\alpha : F^0 \longrightarrow G^0$ (resp. $\alpha : G^0 \longrightarrow F^0$)is an equivalence of functors then it uniquely extends to an equivalence (α_n) of δ-functors.*

Proof: Let β be the inverse of α. There are unique extensions (α_n) of α and (β_n) of β, as given by Theorem 1. Now $\gamma_n = \alpha_n\beta_n$ is an extension of $\alpha\beta = id$. Since identity $\gamma'_n = id$ is also an extension , the uniqueness in Theorem 1 implies that $\gamma_n = \alpha_n\beta_n = id$ for all n. Similarly, $\beta_n\alpha_n = id$ for all n. $\square$

The last lemma indicates that a universal δ-functor can be recovered from F^0. It is natural to ask for a way to compute the δ-functor from F^0. A related question is what functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ are equal to F^0

for some universal δ-functor. The following simple observation gives a necessary condition:

Proposition 1. *If (F^n) is a universal cohomological (homological) δ-functor then F^0 is left exact (resp. right exact).*

Proof: If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence in $\mathcal{C}$ then we have the long exact sequence

$$... \longrightarrow F^{-1}(C) \longrightarrow F^0(A) \longrightarrow F^0(B) \longrightarrow F^0(C) \longrightarrow F^1(A) \longrightarrow ...$$

If (F^n) is cohomological then $F^{-1}(C) = 0$ and we get left exactness, if (F^n) is homological then $F^1(A) = 0$ and we get right exactness. $\square$

Definition 5. *Let (F^n) be a universal cohomological (homological) δ-functor. An object A is called* **acyclic** *(for the δ-functor) if $F^n(A) = 0$ for all $n > 0$ (resp. all $n < 0$).*

We are now going to concentrate on cohomological functors but in parenthesis we will put remarks about analogous statements for homological δ-functors. Note that if $f : X \longrightarrow Y$ is a monomorphism (epimorphism) and Y is acyclic (X is acyclic) then f is an effacing monomorphism (epimorphism). Moreover, suppose that $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence with B acyclic. In the long exact sequence

$$... \longrightarrow F^{k-1}(C) \longrightarrow F^k(A) \longrightarrow F^k(B) \longrightarrow F^k(C) \longrightarrow F^{k+1}(A) \longrightarrow ...$$

we have $F^i(B) = 0$ for all $i > 0$ ($i < 0$ if the δ-functor is homological). It follows that the maps $\delta^k : F^k(C) \longrightarrow F^{k+1}(A)$ are isomorphisms for $k > 0$. This observation often allows to reduce stetements about F^n to statements about F^{n-1} and is called **dimension shifting**. A more refined version of this ideas is the following fundamental

Theorem 2. *Let (F^n) be a cohomological universal δ-functor and set $F = F^0$. Given an exact sequence*

$$0 \longrightarrow A \xrightarrow{f} M^0 \xrightarrow{d^0} M^1 \xrightarrow{d^1} M^2 \xrightarrow{d^2} ...$$

with M^i acyclic for all i, the $k-$th cohomology of the complex

$$(2) \qquad 0 \longrightarrow F(M^0) \xrightarrow{F(d^0)} F(M^1) \xrightarrow{F(d^1)} F(M^2) \xrightarrow{F(d^2)} ...$$

is isomorphic to $F^k(A)$ for all $k \geq 0$.

Similarly, let (F^n) be a homological universal δ-functor and set $F = F^0$. Given an exact sequence

$$0 \longleftarrow A \longleftarrow M_0 \longleftarrow M_1 \longleftarrow M_2 \longleftarrow ...$$

with M_i acyclic for all i, the $k-$th cohomology of the complex

$$0 \longleftarrow F(M_0) \longleftarrow F(M_1) \longleftarrow F(M_2) \longleftarrow ...$$

is isomorphic to $F^{-k}(A)$ for all $k \geq 0$.

Proof: We will give a proof for cohomological functors. We proceed by induction on k.

$\underline{k=0}$: Since the functor F is left exact, the sequence

$$0 \longrightarrow F(A) \longrightarrow F(M^0) \longrightarrow F(M^1)$$

is exact, so $F(A)$ is isomorphic to the kernel $F(d^0)$, i.e. to H^0 of the complex 2.

$\underline{k=1}$: Let A^1 be the image of d^0 (which is canonically isomorphic to the cokernel of f), so we have the following exact sequences:

$$0 \longrightarrow A \longrightarrow M^0 \xrightarrow{d} A^1 \longrightarrow 0$$

and

$$0 \longrightarrow A^1 \xrightarrow{f} M^1 \xrightarrow{d^1} M^2 \xrightarrow{d^2} M^3 \xrightarrow{d^3} ...$$

The long exact sequence associated to the first of these sequences is

$$0 \longrightarrow F(A) \longrightarrow F(M^0) \longrightarrow F(A^1) \longrightarrow F^1(A) \longrightarrow F^1(M^0) = 0 \longrightarrow ...$$

Thus $F^1(A)$ is a cokernel of the morphism $F(d) : F(M^0) \longrightarrow F(A^1)$. On the other hand, left exactness of F implies that the sequence

$$0 \longrightarrow F(A^1) \longrightarrow F(M^1) \longrightarrow F(M^2)$$

is exact, so $F(A^1)$ is a kernel of $F(d^1)$. Thus H^1 of the complex 2 is isomorphic to a cokernel of $F(d)$. Thus $F^1(A)$ and H^1 are canonically isomorphic.

inductive step: Suppose that the result holds for $k < n$, where $n \geq 2$. The long exact sequence associated to

$$0 \longrightarrow A \longrightarrow M^0 \xrightarrow{d} A^1 \longrightarrow 0$$

shows that $F^n(A)$ is isomorphic to $F^{n-1}(A^1)$ (dimension shifting). Since the sequence

$$0 \longrightarrow A^1 \xrightarrow{f} M^1 \xrightarrow{d^1} M^2 \xrightarrow{d^2} M^3 \xrightarrow{d^3} ...$$

is exact, we may use the iductive assumption and conclude that $F^{n-1}(A^1)$ is isomorphic to H^{n-1} of the complex

$$0 \longrightarrow F(M^1) \xrightarrow{F(d^1)} F(M^2) \xrightarrow{F(d^2)} F(M^3) \xrightarrow{F(d^3)} ...$$

which is the same as H^n of the complex 2 (since $n \geq 2$). $\square$

Theorem 2 suggests a method to recover F^n from F^0. The only problem is that we need to use acyclic objects, which are defined by using the functors F^n. Fortunately, we have the following crucial observation

Proposition 2. *Injective (projective) objects are acyclic for any universal cohomological (homological) δ-functor.*

Proof: We give a proof for cohomological δ-functors. Let I be an injective object and $u : I \longrightarrow Y$ an effacing monomorphism. The injectivity of I implies the existence of a morphism $w : Y \longrightarrow I$ such that $wu = id$. Thus, for $n > 0$, we have $id = F^n(id) = F^n(wu) = F^n(w)F^n(u) = F^n(w)0 = 0$, hence $F^n(I) = 0$ (an object is 0 iff $0 = id$). $\square$