Theorem 1 *Let P be a finite p-group and let $Q \lhd P$ be a subgroup of $\gamma_k(P)$.*

1. *If Q is not central in $\gamma_k(P)$ then $|Z(Q)| \geq p^k$ and $|Q| \geq p^{k+1}$.*
2. *If $\gamma_{s+1}(Q) \neq 1$ then $|\gamma_s(Q)/\gamma_{s+1}(Q)| \geq p^k$.*
3. *If Q is not abelian then $|Q| \geq p^{k+2}$ and $|Q/[Q,Q]| \geq p^{k+1}$.*

Proof: Recall that $[\gamma_k(P), \mathfrak{z}_k(P)] = 1$. Since Q is not central in $\gamma_k(P)$, the group Q is not containced in $\mathfrak{z}_k(P)$. Set $Q_i = Q \cap \mathfrak{z}_i(P)$, $i = 1, ..., k$ and let $i < k$. Note that $Q\mathfrak{z}_i(P)/\mathfrak{z}_i(P)$ is a non-trivial normal subgroup of $P/\mathfrak{z}_i(P)$ and $\mathfrak{z}_{i+1}(P)/\mathfrak{z}_k(P)$ is the center of $P/\mathfrak{z}_i(P)$. Thus the groups $\mathfrak{z}_{i+1}(P)/\mathfrak{z}_k(P)$ and $Q\mathfrak{z}_i(P)/\mathfrak{z}_i(P)$ have a non-trivial intersection. In other words, there is $g \in \mathfrak{z}_{i+1}(P) \cap Q\mathfrak{z}_i(P)$ which is not in $\mathfrak{z}_i(P)$. Since $g \in Q\mathfrak{z}_i(P)$ we may write $g = qz$ for some $q \in Q$ and $z \in \mathfrak{z}_i(P)$. Then $q = gz^{-1} \in Q_{i+1}$ and $q \notin Q_i$. This shows that Q_{i+1} strictly contains Q_i for $i < k$. Consequently, $|Q \cap \mathfrak{z}_k(P)| = |Q_k| \geq p^k$. Note that $Q \cap \mathfrak{z}_k(P)$ commutes with $\gamma_k(P)$, hence with Q, i.e. $Q_k \subseteq Z(Q)$. This proves that $|Z(Q)| \geq p^k$. Furthermore, since Q is not contained in $\mathfrak{z}_k(P)$, the group Q_k is a proper subgroup of Q. It follows that $|Q| \geq p|Q_k| \geq p^{k+1}$. This proves 1.

Suppose now that $\gamma_{s+1}(Q) \neq 1$. There is a subgroup M of index p in $\gamma_{s+1}(Q) \neq 1$ and normal in P. Note that Q/M and $\gamma_s(Q/M) = \gamma_s(Q)/M$ are normal subgroups of P/M contained in $\gamma_k(P/M)$. Also, $\gamma_{s+1}(Q/M) = \gamma_{s+1}(Q)/M$ has order p, hence it is not trivial. It follows that $\gamma_s(Q/M)$ is not in the center of $\gamma_k(P/M)$ (otherwise we would have

$$\gamma_{s+1}(Q/M) = [\gamma_s(Q/M), Q/M] \subseteq [\gamma_s(Q/M), \gamma_k(P/M)] = 1,$$

a contradiction). By part 1. we conclude that $|\gamma_s(Q/M)| \geq p^{k+1}$. Since $|\gamma_s(Q/M)| = p|\gamma_s(Q)/\gamma_{s+1}(Q)|$, part 2. follows.

Assume now that Q is not abelian. Then $[Q : Z(Q)] \geq p^2$. since $|Z(Q)| \geq p^k$ by part 1., we see that $|Q| \geq p^{k+2}$. This shows the first claim of 3. Note that $[Q, Q]$ has a subgroup N of index p which is normal in P. Thus Q/N is a non-abelian normal subgroup of P/N contained in $\gamma_k(P/N)$. Thus $|Q/N| \geq p^{k+2}$. Since $|Q/N| = p|Q/[Q,Q]|$, the second claim in 3. follows.