Homework 1

due on Monday, September 19

Problem 1. Let G be a group with a normal subgroup N and a (not necessarily normal) subgroup H. Suppose that N has a chain of normal (in N) subgroups

$$N_1 = N \geq N_2 \geq N_3 \geq ...$$

such that $[H, N_i] \subseteq N_{i+1}$ for all i. Prove that $[\gamma_i(H), N_j] \subseteq N_{i+j}$ for all i, j.

Problem 2. Let P be a finite p-group.

a) Prove that if $\gamma_2(P) \cap \mathfrak{z}_2(P)$ is cyclic then $\gamma_2(P)$ is cyclic.

b) If the center of $[P, P]$ is cyclic then $[P, P]$ is cyclic.

c) If the center of $\mathrm{Frat}(P)$ is cyclic then $\mathrm{Frat}(P)$ is cyclic.

Problem 3. Let P be a finite p-group such that $[P^{(1)} : P^{(2)}] \leq p^2$. Prove that the commutator subgroup $P^{(1)}$ of P is abelian.

Problem 4. The lower p-central series of G is the descending central series

$$G = \lambda_1(G) \geq \lambda_2(G) \geq \lambda_3(G) \geq ...$$

of subgroups of G, where $\lambda_{i+1}(G) = [\lambda_i(G), G]\lambda_i(G)^p$ for all i. Prove that

a) If $G = G_1 \geq G_2 \geq G_3 \geq ...$ is a descending central series such that G_i/G_{i+1} has exponent p for all i, then $\lambda_i(G) \subseteq G_i$ for all i.

b) $[\lambda_i(G), \lambda_j(G)] \leq \lambda_{i+j}(G)$ for all i, j.

c) If $\lambda_2(G) = \gamma_2(G)$ then $\lambda_i(G) = \gamma_i(G)$ for all i.

Problem 5. Prove that if a finite p-group has an abelian subgroup of index p^2 then it has a normal abelian subgroup of index p^2.

Problem 6. Let p be a prime and let $\mathbb{F}_p$ be the field with p-elements. Find the lower central series and the derived series of the group of $n \times n$ upper-triangular matrices over $\mathbb{F}_p$ with all diagonal entries equal to 1.