Homework 3

Due on Wednesday, October 19

Problem 1. Let P be a regular 3-group. Prove that $\gamma_n(P) \subseteq \gamma_2(P)^{3^{n-2}}$ for every $n \geq 3$.

Correction This is false! It is true that $\gamma_4(P) \subseteq \gamma_2(P)^3$ for a regular 3-gropup P, so perhaps $\gamma_n(P) \subseteq \gamma_2(P)^{3^{n-3}}$ for every $n \geq 3$. I do not know a counterexample at the moment, but I am doubftul if this is true. The origianl problem is from Hupperts book "Endliche gruppen I", page 335, Aufgaben 26 (Chapter III, end of section 10).

Problem 2. Let P, Q be regular p-groups such that $\gamma_2(P)$ has exponent p. Prove that $P \times Q$ is regular.

Problem 3. Let G be a group such that $[G, G]$ is abelian. Prove that

$$[x, y^n] = [x, y]^{\binom{n}{1}}[x, y, y]^{\binom{n}{2}}...[x, y, ..., y]^{\binom{n}{n}}$$

for any $x, y \in G$ and any positive integer n.

Problem 4. Let P be a regular p-group such that $[P, P]$ has exponent p^k. Prove that if $p^k|(n-1)$ then the map $p \mapsto p^n$ is an automorphism of P.

Problem 5. Let P be a regular p-group. Suppose that the map $p \mapsto p^n$ is an automorphism of P and let p^k be the highest power of p which divides $n-1$.

a) Show that P^{p^k} is contained in the center of P.

b) Show that $[P, P] \subseteq \Omega_k(P)$.

c)* Show that b) holds without the assumption that P is regular.

Problem 6. Let P be a finite p-group of order p^5, $p \geq 5$. Prove that $[P, P]$ can-not be isomorphic to $\mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.