Homework 3

Due on Wednesday, October 19

Problem 1. Let P be a regular 3-group. Prove that $\gamma_n(P) \subseteq \gamma_2(P)^{3^{n-2}}$ for every $n \geq 3$.

Solution. This is false in general. There exist 3-groups P of exponent 3 such that $\gamma_3(P)$ is not trivial. Any such group is regular and satisfies $\gamma_2(P)^3 = 1$, so it is a counterexample to the problem. For an explicit example consider the group of order 3^7 given by the presentation

$$< x, y, z : x^3 = y^3 = z^3 = [x, y]^3 = [y, z]^3 = [z, x]^3 = 1, [x, y, z] = [z, x, y] = [y, z, x] > .$$

This is the Burnside group $B(3, 3)$, i.e. the quotient of the free group F on 3 generators by its subgroup F^3.

It can be proved that if P is a 3-group of exponent 3 then $\gamma_4(P) = 1$. With more effort one can show that $\gamma_4(P) \subseteq \gamma_2(P)^3$ for any regular 3-group P. Now, suppose that P is regular and $\gamma_{2m}(P) \subseteq \gamma_2(P)^{3^{m-1}}$ for some $m \geq 1$. Then

$$\gamma_{2m+1}(P) = [\gamma_{2m}(P), P] \subseteq [\gamma_2(P)^{3^{m-1}}, P] = [\gamma_2(P), P]^{3^{m-1}} = \gamma_3(P)^{3^{m-1}}$$

and

$$\gamma_{2m+2}(P) = [\gamma_{2m+1}(P), P] \subseteq [\gamma_3(P)^{3^{m-1}}, P] = [\gamma_3(P), P]^{3^{m-1}} = \gamma_4(P)^{3^{m-1}} \subseteq \gamma_2(P)^{3^m}.$$

Thus we see by easy induction that $\gamma_{2n}(P) \subseteq \gamma_2(P)^{3^{n-1}}$ for every $n \geq 1$ and every regular 3-group P. Perhaps this is what Huppert intended to put in his book.

Problem 2. Let P, Q be regular p-groups such that $\gamma_2(P)$ has exponent p. Prove that $P \times Q$ is regular.

Solution. Since P is regular and $\gamma_2(P)$ is of exponent p we have $(ab)^p = a^p b^p$ for any $a, b \in P$. Given (a, x), (b, y) in $P \times Q$ let H be the subgroup generated by these elements. The projection $P \times Q \longrightarrow Q$ maps H onto the subgroup $B =< x, y >$ of Q generated by x, y. Thus this projection maps $\gamma_2(H)$ onto $\gamma_2(B)$. In particular, given $c \in \gamma_2(B)$, there is $d \in P$ such that $(d, c) \in \gamma_2(H)$. Looking at the projection $P \times Q \longrightarrow P$ we see that $d \in \gamma_2(P)$.

Since Q is regular, we have $(xy)^p = x^p y^p c^p$ for some $c \in \gamma_2(B)$. As we observed, there is $d \in \gamma_2(P)$ such that $(d, c) \in \gamma_2(H)$. Note that $(d, c)^p = (d^p, c^p) = (1, c^p)$. Thus

$$((a, x)(b, y))^p = ((ab)^p, (xy)^p) = (a^p b^p, x^p y^p c^p) = (a^p, x^p)(b^p, y^p)(1, c^p) = (a, x)^p (b, y)^p (d, c)^p.$$

This confirms that $P \times Q$ is regular.

Problem 3. Let G be a group such that $[G, G]$ is abelian. Prove that

$$[x, y^n] = [x, y]^{\binom{n}{1}} [x, y, y]^{\binom{n}{2}} ... [x, y, ..., y]^{\binom{n}{n}}$$

for any $x, y \in G$ and any positive integer n.

Solution. Recall that $[x, ab] = [x, b][x, a]^b$ and $y^b = y[y, b]$. We use induction on n. For $n = 1$ the result clearly true. Assuming it holds for n we have

$$[x, y^{n+1}] = [x, y^n y] = [x, y][x, y^n]^y = [x, y]([x, y]^{\binom{n}{1}} [x, y, y]^{\binom{n}{2}} ... [x, y, ..., y]^{\binom{n}{n}})^y =$$

$$= [x, y]([x, y]^y)^{\binom{n}{1}} ([x, y, y]^y)^{\binom{n}{2}} ... ([x, y, ..., y]^y)^{\binom{n}{n}} =$$

$$= [x,y]([x,y][x,y,y])^{\binom{n}{1}}([x,y,y][x,y,y,y])^{\binom{n}{2}}...([x,y,...,y][x,y,...,y,y])^{\binom{n}{n}}) =$$
$$= [x,y]^{1+\binom{n}{1}}[x,y,y]^{\binom{n}{1}+\binom{n}{2}}...[x,y,...,y]^{\binom{n}{n-1}+\binom{n}{n}}[x,y,...,y,y]^{\binom{n}{n}} =$$
$$= [x,y]^{\binom{n+1}{1}}[x,y,y]^{\binom{n+1}{2}}...[x,y,...,y]^{\binom{n+1}{n+1}}.$$

Problem 4. Let P be a regular p-group such that $[P,P]$ has exponent p^k. Prove that if $p^k|(n-1)$ then the map $p \mapsto p^n$ is an automorphism of P.

Solution. Write $n = 1 + p^k m$. We need to prove that $(ab)^n = a^n b^n$ for any $a, b \in P$. Note that $(ab)^n = a(ba)^{n-1}b$, so it suffices to show that $(ba)^{n-1} = a^{n-1}b^{n-1}$.

Since P is regular, we have $(ba)^{p^k} = b^{p^k}a^{p^k}c^{p^k}$ for some $c \in \gamma_2(< a, b >)$. But p^k is the exponent of $\gamma_2(P)$ so $c^{p^k} = 1$ and $(ba)^{p^k} = b^{p^k}a^{p^k}$. Furthermore, $[x^{p^k}, y] = [x,y]^{p^k} = 1$ for any $x, y \in P$. It follows that P^{p^k} is central in P. Thus

$$(ba)^{n-1} = (ba)^{p^k m} = (b^{p^k}a^{p^k})^m = a^{p^k m}b^{p^k m} = a^{n-1}b^{n-1}.$$

Problem 5. Let P be a regular p-group. Suppose that the map $a \mapsto a^n$ is an automorphism of P and let p^k be the highest power of p which divides $n-1$.

a) Show that P^{p^k} is contained in the center of P.

Solution. Let $n - 1 = p^k m$, where $p \nmid m$. Denote the automorphism $a \mapsto a^n$ by ϕ (so $\phi(a) = a^n$). We have

$$b^{-1}\phi(a)b = b^{-1}a^n b = (b^{-1}ab)^n = \phi(b^{-1}ab) = \phi(b)^{-1}\phi(a)\phi(b) = b^{-n}\phi(a)b^n$$

for any $a, b \in P$. Thus $b^{1-n}\phi(a)b^{n-1} = \phi(a)$ for all $a, b \in P$. Since ϕ is surjective, b^{n-1} is central for all $b \in P$. Recall that $n-1 = p^k m$, where $p \nmid m$. Since P is a p-group, we have b^{p^k} is central for all $b \in P$.

Remark. Note that we have not used here the assumption that P is regular.

b) Show that $[P,P] \subseteq \Omega_k(P)$.

Solution. Recall that we proved in class that in a regular p-group P the inclusion $\gamma_{i+1}(P) \subseteq \Omega_k(P)$ is equivalent to $P^{p^k} \subseteq \mathfrak{z}_i(P)$. Since we showed in a) that $P^{p^k} \subseteq \mathfrak{z}_1(P)$, we see that $[P,P] \subseteq \Omega_k(P)$.

c) Show that b) holds without the assumption that P is regular.

Solution. We have seen in a) that $\phi(b)b^{-1}$ is central for all $b \in P$. Given $a, b \in P$ we have $\phi(a) = au$, $\phi(b) = bw$ for some central elements u, w. Thus

$$\phi([a,b]) = [\phi(a), \phi(b)] = [au, bw] = [a,b].$$

It follows that $x^n = \phi(x) = x$ for all $x \in [P,P]$, i.e. $x^{n-1} = 1$ for all $x \in [P,P]$. This implies that $x^{p^k} = 1$ for all $x \in [P,P]$, i.e. $[P,P] \subseteq \Omega_{\{k\}}(P)$.

Remark. It is known that if G is a group and f is an automorphism of G such that $f(g)$ is a power of g for every element $g \in G$ then $f(g)g^{-1}$ is central for every $g \in G$ and trivial for $g \in [G,G]$. This result is usually called *Cooper's Theorem*.

Problem 6. Let P be a finite p-group of order p^5, $p \geq 5$. Prove that $[P,P]$ can-not be isomorphic to $\mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

Solution. P is a regular p-group. Suppose that $[P,P]$ is isomorphic to $\mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Thus $[P,P] = \mathrm{Frat}P$ and $P/\mathrm{Frat}P$ has order p^2. It follows that P is 2-generated and therefore $\gamma_2(P)/\gamma_3(P)$ is cyclic. If $\Omega_1(P)$ had order $\geq p^3$ then $P/\Omega_1(P)$ would be abelian and therefore $[P,P] \subseteq \Omega_1(P)$, which is false. Thus $\Omega_1(P)$ has order p^2 and therefore P^p has order p^3. Since $P^p \subseteq \mathrm{Frat}P$, we have $[P,P] = \mathrm{Frat}P = P^p$. It follows that $\gamma_3(P) = [[P,P],P] = [P^p,P] = [P,P]^p$ so $\gamma_2(P)/\gamma_3(P) = [P,P]/[P,P]^p$ is not cyclic, a contradiction.