Theorem 1. *Let X be an affine irreducible variety of dimension d, $f \in k[X]$ a non-zero function and $X_f = \{x \in X : f(x) = 0\}$. Then either $X_f = \emptyset$ or $\dim X_f = d - 1$.*

Proof: Let $\phi : X \longrightarrow \mathbb{A}^d$ be a finite surjective morphism. The morphism

$$\psi : X \longrightarrow \mathbb{A}^d \times \mathbb{A} = \mathbb{A}^{d+1}, \quad \psi(x) = (\phi(x), f(x))$$

is finite. Indeed, if $\pi : \mathbb{A}^d \times \mathbb{A} \longrightarrow \mathbb{A}^d$ is the projection, then $\phi = \pi\psi$, so the claim follows from the following simple exercise:

Exercise: If $g : X \longrightarrow Y$ and $h : Y \longrightarrow Z$ are morphisms of **affine** varieties and hg is finite then g is finite.

Thus the image $\psi(X)$ is an irreducible closed subset of $\mathbb{A}^{d+1}$ of dimension d. It follows that $\psi(X) = V(h) = \{x \in \mathbb{A}^{d+1} : h(x) = 0\}$ for some irreducible polynomial $h(x_1, ..., x_{d+1}) = \sum_{i=0}^{s} a_i(x_1, ..., x_d)x_{d+1}^i$. Note that $a_0 \neq 0$ (otherwise $h = X_{d+1}$ and $\psi(X) \subseteq \mathbb{A}^d \times \{0\}$, i.e. $f = 0$, which is false).

Exercise: Use the fact that f is integral over $k[\mathbb{A}^d]$ to show that a_s is constant.

Note that if $w \in X_f$ then $\phi(w) = (x_1, ..., x_d)$, $\psi(w) = (x_1, ..., x_d, 0)$ an $h(\psi(w)) = 0$. Thus $a_0(x_1, ..., x_d) = 0$. Conversely, suppose that $a_0(x_1, ..., x_d) = 0$. Then $h(x_1, ..., x_d, 0) = 0$ so $(x_1, ..., x_d, 0) \in \psi(X)$, i.e there is $w \in X$ such that $(\phi(x), f(x)) = (x_1, ..., x_d, 0)$. We see that $w \in X_f$ and $\phi(w) = (x_1, ..., x_d)$. This shows that $\phi(X_f) = \{x \in \mathbb{A}^d : a_0(x) = 0\}$. It follows that if $X_f \neq \emptyset$ then $\phi(X_f)$ is not empty and therefore is has dimension $d - 1$ (being defined by a single equation in $\mathbb{A}^d$). Since $\phi : X_f \longrightarrow \phi(X_f)$ is finite surjective, $\dim X_f = d - 1$. □

Exercise: Show that if $x = (x_1, ...x_d) \in \mathbb{A}^d$ and $t \in k$ is such that $h(x_1, ..., x_d, t) = 0$ then there is $w \in X$ such that $\phi(w) = x$ and $f(w) = t$.

Exercise: Use Theorem 1 to show that if $X \subseteq \mathbb{P}^n$ is closed of dimension d and $F \in k[x_0, ..., x_n]$ is a homogeneous polynomial (of positive degree) which is not identically 0 on any component of X then $\dim X_F = d-1$, where $X_F = \{x \in X : F(x) = 0\}$ (in particular, X_F is non-empty if $d > 0$).

Theorem 2. *Let X be an irreducible algebraic variety of dimension d and let $f \in \mathcal{O}_X(X)$ be a regular function on X. Then any (non-empty) component of $X_f = \{x \in X : f(x) = 0\}$ has dimension $d - 1$.*

Proof: Let Z be an irreducible component of X_f (assuming X_f is non-empty). Let U be an affine open subset of X such that $U \cap Z$ is not empty. Thus $Y = U \cap Z$ is an irreducible component of $U \cap X_f = U_f$ and $\dim Y = \dim Z$, $\dim U = \dim X$. Let V be the union of all irreducible components of U_f except Y. There is a regular function $h \in k[U]$ such that h vanishes on V but not on Y. The set $D(h) = \{x \in U : h(x) \neq 0\}$ is an affine open set in U (hence in X) such that $D(h)_f = D(h) \cap X_f = D(h) \cap Y$ is not empty. Thus $\dim D(h) = \dim X = d$ and $\dim D(h)_f = \dim Y = \dim Z$ (since it is a non-empty open subset of Z). By Theorem 1, $\dim D(h)_f = \dim D(h) - 1$, i.e. $\dim Z = d - 1$. □

Corollary 1. *If $X \subseteq \mathbb{P}^n$ is closed irreducible of dimension d and $F \in k[x_0, ..., x_n]$ is a homogeneous polynomial (of positive degree) which is not identically 0 on X then every component of $X_F = \{x \in X : F(x) = 0\}$ has dimension $d - 1$.*

Proof: Let Z be a component of X_F and let $[z_0, ..., z_n]$ be a point of Z. Pick i such that $z_i \neq 0$ and let U be the affine open subset of X given by $x_i \neq 0$. Then U is irreducible and $Z \cap U$ is a non-empty component of U_f, where $f = F/x_i^{\deg F}$ is a non-zero function on U. Thus $\dim(Z \cap U) = \dim U - 1$ by Theorem 2. Since $\dim(Z \cap U) = \dim Z$ and $\dim U = \dim X$, the result follows. □

Corollary 2. *If X is an irreducible variety of dimension d and let $f_1, ..., f_k \in \mathcal{O}_X(X)$ be regular functions on X. Then any (non-empty) component of $Y = \{x \in X : f_1(x) = f_2(x) = ... = f_k(x) = 0\}$ has dimension $\geq d - k$.*

Proof: Note that a component of Y is a component of Z_{f_k} for some component Z of $\{x \in X : f_1(x) = f_2(x) = ... = f_{k-1}(x) = 0\}$ and use induction. □

Corollary 3. *If $X \subseteq \mathbb{P}^n$ is locally closed irreducible of dimension d and $F_1, ..., F_k \in k[x_0, ..., x_n]$ are homogeneous polynomials (of positive degree) then every (non-empty) component of $Y = \{x \in X : F_1(x) = ... = F_k(x) = 0\}$ has dimension $\geq d - k$.*

Proof: Exercise. Use Corollary 2 and the ideas from proof of Corollary 1. □

Theorem 3. *Let X, Y be locally closed irreducible subvarieties of $\mathbb{P}^N$ of dimensions m, n respectively. Then every (non-empty) irreducible component of $X \cap Y$ has dimension $\geq n + m - N$. If X, Y are closed and $n + m \geq N$ then $X \cap Y \neq \emptyset$.*

Proof: Let Z be a component of $X \cap Y$ and let $z \in Z$. Let U be an affine open subset of $\mathbb{P}^N$ containing z and isomorphic to $\mathbb{A}^N$. Then $U \cap Z$ is a non-empty component of $(U \cap X) \cap (U \cap Y)$. Since the dimensions of $U, U \cap X, U \cap Y, U \cap Z$ coincide with the dimensions of $\mathbb{P}^N, X, Y, Z$ respectively, we may assume that X, Y are inside $\mathbb{A}^N$. Note that $X \cap Y$ is isomorphic to $(X \times Y) \cap \Delta$, where Δ is the diagonal in $\mathbb{A}^N \times \mathbb{A}^N$. If $x_1, ..., x_N, y_1, ..., y_N$ are coordinates on $\mathbb{A}^N \times \mathbb{A}^N$ then Δ is given by zeros of N regular functions $x_i - y_i$, $i = 1, 2, ..., N$. By Corollary 2, each component of $(X \times Y) \cap \Delta$ has dimension $\geq n + m - N$. This proves the first part of the theorem.

In order to show the second part, consider the cones C_X, C_Y of X and Y in $\mathbb{A}^{N+1}$. These are closed irreducible subset of dimension $m + 1$, $n + 1$ respectively and both contain the point $(0, 0, ..., 0)$. By the first part of the theorem, any component of $C_X \cap C_Y$ which contains $(0, 0, ..., 0)$ has dimension $\geq (m + 1) + (n + 1) - (N + 1) = m + n + 1 - N > 0$. Thus there is a non-zero point in this component and its image in $\mathbb{P}^N$ is a point of $X \cap Y$. □

Remark. One may be tempted to replace $\mathbb{P}^N$ in Theorem 3 by an irreducible variety W of dimension N. But the results does not hold in such a generality. Consider the quadric W in $\mathbb{P}^3$ given by $X_0^2 + X_1^2 + X_2^2 + X_3^2 = 0$. This is an irreducible subset of dimension 2. Consider the morphisms $F_e : \mathbb{P}^1 \longrightarrow W$ given by $F_e([t_0, t_1]) = [t_0, eit_0, t_1, eit_1]$, where $e = \pm 1$ and $i^2 = -1$. The images of F_{-1}, F_1 are closed irreducible curves on W with empty intersection. This disproves the second part of the theorem. Passing to cones in $\mathbb{A}^4$ gives a counterexample to the first part. Note that W is birational to $\mathbb{P}^2$ but our consideration shows that it is not isomorphic to $\mathbb{P}^2$.

In the proof of Theorem 3 we used the following two simple, but useful facts.

Exercise. Let X, Y be varieties of dimension m, n respectively. Then the dimension of $X \times Y$ is $m + n$. **Hint.** Reduce to the case of affine varieties and then find a finite surjective map onto $\mathbb{A}^{m+n}$ (for a different proof consult the book).

Exercise. Let $X \subseteq \mathbb{P}^N$ be closed subset of dimension d and let C_X be the (closed) cone of X in $\mathbb{A}^N$. Show that $\dim C_X = d+1$. **Hint**. Reduce to irreducible X, so C_X is irreducible. By the discussion below there is a chain X_i, $i = 0, 1, ..., N$ of irreducible closed subsets of $\mathbb{P}^N$ such that $\dim X_i = i$ and $X_d = X$. Set $Z_0 = \{(0, 0, ...0)\}$ and $Z_{i+1} = C_{X_i}$. Then Z_i form a chain of irreducible, closed subsets of $\mathbb{A}^{N+1}$ of length $N+1$. Since $\dim A^{N+1} = N+1$, we see that $dim Z_i = i$. Since $Z_{d+1} = C_X$, the result follows.

We have seen that $\dim X$ is equal to the largest n such that X has a chain $X_0 \subsetneq X_1 \subsetneq ... \subsetneq X_n \subseteq X$ such that each X_i is closed and irreducible. We will show now that if X is irreducible then every chain of irreducible closed subsets of X can be refined to a chain of length $\dim X$ (i.e. all maximal such chains have length $\dim X$). For this it suffices to show the following

Proposition 1. *Let X be irreducible and let Y be a proper irreducible closed subset of X. Then $Y \subseteq Z \subseteq X$ for some closed irreducible subset Z of dimension* $\dim X - 1$.

Proof: Let U be an affine open subset of X such that $V = U \cap Y$ is not empty. Thus U is an irreducible affine variety and V is a non-empty proper closed irreducible subset of V. Thus there is $0 \neq f \in k[U]$ such that $V \subseteq U_f$. Since V is irreducible, there is an irreducible component W of U_f containing V. By Theorem 2, $\dim W = \dim U - 1 = \dim X - 1$. Let Z be the closure of W in X. Then Z is irreducible, contains Y (since V is dense in Y), and $Z \cap U = W$, so W is a non-empty open subset of Z and therefore $\dim Z = \dim W = \dim X - 1$. □

In the case when X is affine our obeservation translates into the following algebraic statement:

If R is a domain finitely generated over an algebraically closed field k then all maximal chains of prime ideals in R have the same length equal to the Krull dimension of R.

Not every Noetherian ring has the above property and Noetherian rings having the property are called **catenary**. Practically all Noetherian rings one encounters in algebraic geometry are catenary though. The translation of Theorem 2 for affine X to a statement about finitely generated $k-$algebras holds for all catenary rings (Exercise: formulate this result), and the following slightly weaker version, usually called Krull's Hauptidealsatz or Principal Ideal Theorem, holds for all Noetherian rings and it is a cornerstone of the theory of Krull dimension

Principal Ideal Theorem: *Let R be a Noetherian domain and $0 \neq f \in R$ a non-unit. Then every prime ideal of R which is minimal among all prime ideals containing f is also minimal among all non-trivial prime ideals of R.*

Exercise: Derive this result for $R = k[X]$ from Theorem 2.

All the results we proved about dimension of algebraic varieties were derived from the Noether normalization theorem. Let us formulate here a strong version of this result

Strong Noether Normalization: *Let k be a field and let R be a finitely generated k-algebra of dimension d. Suppose that $I_1 \subset I_2 \subset ... \subset I_m$ is a chain of ideals of R such that the sequence $d_i = \dim(R/I_i)$ is strictly decreasing. Then R contains a subring $k[x_1, ..., x_d]$ with $x_1, ..., x_d$ algebraically independent over k and such that R is a finitely generated $k[x_1, ..., x_d]$-module and the ideal $I_j \cap k[x_1, .., x_d]$ is generated by $x_{d_j+1}, ..., x_d$ for*

$j = 1, 2, ..., m$. *If R is graded (i.e. it is the graded ring of a projective veriety) and the ideals I_j are homogeneous then the x_i can be chosen homogeneous.*

We leave it as a challenging exercise to prove this result and to formulate it in the langauge of affine/projective varieties when k is algebraically closed.

Theorem 4. *Let $f : X \longrightarrow Y$ be a dominant morphism of irreducible algebraic varieties. Then*

(1) $n = \dim X \geq m = \dim Y$;
(2) *every non-empty component of $f^{-1}(y)$ has dimension $\geq n - m$ for every $y \in Y$;*
(3) *there is an open dense set $V \subseteq Y$ such that $\dim f^{-1}(y) \leq n - m$ (i.e. either f^{-1} is empty or $\dim f^{-1}(y) = n - m$) for every $y \in V$;*

Proof: It suffices to show the theorem in the case when Y is affine (all statements are local, i.e. if they hold for each member of an open cover of Y then they hold for Y). If U is a non-empty affine open set in X then U is dense in X and $f(U)$ is dense in Y,i.e. the morphism $f : U \longrightarrow Y$ is dominant. In other words, $k[Y]$ embeds into $k[U]$ and therefore $k(Y)$ is a subfield of $k(U) = k(X)$. It follows that $n \geq m$. This proves (1).

Let g be a finite surjective morphism $g : Y \longrightarrow \mathbb{A}^m$ and $h = gf$. For a point $y \in Y$ and $g(y) = a = (a_1, ..., a_m)$ the fiber $h^{-1}(a)$ is the union of fibers of f over the finite set $g^{-1}(a)$. Thus every component of $f^{-1}(y)$ is a component of $h^{-1}(a)$. Thus it suffices to show (2) and (3) for h. In other words, we may assume that $Y = \mathbb{A}^m$. Now if $x_1, ..., x_m$ are the coordinate functions on $\mathbb{A}^m$, then $\{a\}$ is given by zeros of the m functions $f_i = x_i - a_i$, $i = 1, 2, ..., m$. Thus $h^{-1}(a)$ is the set of zeros of the regular on X functions $f_i h$. By Corollary 2, every non-empty component of $h^{-1}(a)$ has dimension $\geq n - m$. This proves (2).

Let $U_1, ..., U_s$ be a finite affine open cover of X. In order to prove (3) it suffices to show that there are non-empty open sets V_i in Y such that for $y \in V_i$ we have $\dim(f^{-1}(y) \cap U_i) \leq n - m$. In fact, then $V = \bigcap V_i$ is non-empty open in Y and if $y \in Y$ and $Z \neq \emptyset$ is a component of $f^{-1}(y)$ then $Z \cap U_i \neq \emptyset$ for some i so $\dim Z = \dim(Z \cap U_i) \leq \dim(f^{-1}(y) \cap U_i) \leq n - m$. Since $\dim Z \geq n - m$ by (2), we get $\dim Z = n - m$. Thus $\dim f^{-1}(y) = n - m$ for $y \in V \cap f(X)$.

Let then U be an affine open set in X and consider $f : U \longrightarrow Y = \mathbb{A}^m$. Since f is dominant, we can consider $k[Y] = k[x_1, ..., x_m]$ as a subring of $k[U]$. We have $k[U] = k[v_1, ..., v_t]$ for some regular functions v_i on U. Since the transcendence degree of $k(U)$ over $k(Y)$ is $n - m$, for any subset $T = \{w_1, ..., w_{n-m+1}\}$ of size $n - m + 1$ of $\{v_1, .., v_t\}$ there is a non-zero polynomials $P_T \in k[Y][T_1, ..., T_{n-m+1}]$ such that $P_[T](w_1, ..., w_{n-m+1}) = 0$. Let $R_T = R_T(x_1, ..., x_m) \in k[Y]$ be a non-zero coefficient of P_T. The set $V_U = \{y \in Y : R_T(y) \neq 0 \text{ for all subsets } T\}$ is an open dense subset of Y. Let $y \in V_U$. Any component Z of the fiber $f^{-1}(y)$ is a closed irreducible subset of U (if the fiber is empty, then there is nothing to prove). We have $k[Z] = k[\overline{v}_1, ..., \overline{v}_t]$, where $\overline{v}_i$ is the restriction of v_i to Z. Note that the restriction of P_T to Z is a non-zero polynomial $\overline{P}_T$ in $k[T_1, ..., T_{n-m+1}]$. Clearly, $\overline{P}_T(\overline{w}_1, ..., \overline{w}_{n-m+1}) = 0$. It follows that any $n - m + 1$ elements of of the set $\{\overline{v}_1, ..., \overline{v}_t\}$ are algebraically dependent over k. Thus the transcendence degree of $k(Z)/k$ is $< n - m + 1$, i.e. $\dim Z \leq n - m$. This shows that $\dim(f^{-1}(y) \cap U) \leq n - m$ for $y \in V_U$. This completes our proof of (3). □

Definition 1. *For an algebraic variety X and a point $x \in X$ define $\dim_x X$ as the maximum of the dimensions of all components of X which contain x.*

Theorem 5. *Let $f : X \longrightarrow Y$ be a morphism of algebraic varieties. For each k the set $X_k = \{x \in X : \dim_x f^{-1}(f(x)) \geq k\}$ is closed.*

Proof: Our proof is by induction on $\dim X$. If $\dim X = 0$ then the result is clear. Suppose the result holds for varieties of dimension $< n$ and consider X of dimension n. First note that it suffices to show the theorem for irreducible X. In fact, since every component of the fiber $f^{-1}(f(x))$ is contained in a component of X, X_k is the union of the sets Z_k for all irreducible components Z of X. Also, we may assume that f is dominant (replacing Y by the closure of $f(X)$). Let $d = \dim X - \dim Y$. If $k \leq d$ then $X_k = X$ by (2) of Theorem 4. Suppose that $k > d$. By (3) of Theorem 4, there is a non-empty open set U in Y such that $\dim f^{-1}(y) \leq d$ for $y \in U$. It follows that $f^{-1}(U) \cap Y_k = \emptyset$. Since f is dominant, $f^{-1}(U)$ is a non-empty open set so the complement Z of $f^{-1}(U)$ is a proper closed set in X containing X_k. Thus Since X is irreducible, we have $\dim Z < n$. Apply now the inductive assumption to the morphism $f : Z \longrightarrow Y$ and note that $Z_k = X_k$ to get that X_k is closed in Z, hence in X. $\Box$

Corollary 4. *Suppose that $f : X \longrightarrow Y$ is closed. Then for each k the set $Y_k = \{y \in Y : \dim f^{-1}(y) \geq k\}$ is closed.*

Proof: Note that $Y_k = f(X_k)$. Since X_k is closed by Theorem 5 and f is a closed map, Y_k is closed. $\Box$

Remark. Note that if X is a projective variety, then any f is closed.

Let us remark that (3) of Theorem 4 can be strengthened: there is an open set V such that the fibers $f^{-1}(y)$ have dimension $n - m$ (i.e. are non-empty) for all $y \in V$. This follows from a fact that the image of a dominant morphism contains a non-empty open set. This can be proved from what we know about finite morphisms (as is done in Schafarevich), but we prove here a stronger result, which relies on Noether normalization over non-algebraically closed fields (which we did not prove). It is a nice illustration why results over non-algebraically closed fields are needed even in algebraic geometry over algebraically closed fields.

Recall the version of Noether normalization we need:

Noether Normalization: *Let k be a field and let R be a finitely generated k-algebra. Then R contains a subring $k[x_1, ..., x_d]$ with $x_1, ..., x_d$ algebraically independent over k and such that R is a finitely generated $k[x_1, ..., x_d]$-module.*

Theorem 6. *Let $f : X \longrightarrow Y$ be a dominant morphism of affine varieties with Y irreducible. There is a non-empty open subset V in Y and a finite surjective morphism $g : f^{-1}(V) \longrightarrow V \times \mathbb{A}^d$, $d = \dim X - \dim Y$, such that $f = \pi g$, where $\pi : V \times \mathbb{A}^d \longrightarrow V$ is the projection.*

Proof: Since f is dominant, we can consider $k[Y]$ as a subring of $k[X]$. Let $F = k(Y)$ and let R be the the ring obtained from $k[X]$ by inverting all non-zero elements of $k[Y]$. Then R is a finitely generated F-algebra. Note that we have a natural homomorphism $\eta : k[X] \longrightarrow R$ (which does not have to be injective, but is injective on k[Y]) and every element of R is of the form $\eta(r)/\eta(g)$, where $r \in k[X]$ and $g \in k[Y]$. If $\eta(r) = 0$ the there is a non-zero $a \in k[Y]$ such that $ar = 0$ (we are using here basic facts about localization). Since R is Noetherian, the kernel of η is finitely generated and there is a non-zero $a \in k[Y]$ such that $\eta(r) = 0$ iff $ar = 0$.

By Noether normalization, there are elements $z_1, ..., z_d \in R$, algebraically independent over F such that R is a finitely generated $F[z_1, ..., z_d]$-module. We mays assume that $z_i = \eta(x_i)$ for $x_i \in k[X]$. Recall now that $k[X] = k[v_1, ..., v_t]$ for some functions v_i. For each i there is a monic polynomial $P_i(T) \in F[z_1, ..., z_d][T]$ such that $P_i(\eta(v_i)) = 0$. The coefficients of the polynomials P_i involve a finite number of elements of F. Thus we may consider these polynomials as members of $k[Y][1/h][x_1, ..., x_d][T]$ for some $h \in k[Y]$ which can be assumed divisible by a. Then $P_i(v_i) = 0$ in $k[X][1/h]$. This shows that $k[X][1/h]$ is a finitely generated $k[Y][1/h][x_1, ..., x_d]$-module. Now if $V = D(h) = \{y \in Y : h(y) \neq 0\}$ then $k[V] = k[Y][1/h]$, $k[f^{-1}(V)] = k[X][1/h]$, $k[Y][1/h][x_1, ..., x_d] = k[V \times \mathbb{A}^d]$ and the natural homomorphisms

$$k[V] \longrightarrow k[V \times \mathbb{A}^d] \longrightarrow k[f^{-1}(V)]$$

correspond to morphisms of varieties as claimed in the theorem. $\square$.

Exercise: Derive Theorem 4 and the fact that image of a dominant morphism contains a non-empty open set from Theorem 6.