Homework 10
due on Monday, August 1

Read carefully sections 1,2 of Chapter 2 in Shen's book and and sections 1.6, 1.7, 1.11 in Stroll's book. Solve the following problems.

Problem 1. Let R be a relation on a set A so R is subsets of $A \times A$. Define a new relation R^{-1} as follows: $R^{-1} = \{(a, b) \in A \times A : (b, a) \in R\}$. In other words, $aR^{-1}b$ if and only of bRa.

a) Prove that if R is a partial order then so is R^{-1}. Prove that if R is linear then so is R^{-1}.

b) Prove that R is symmetric if and only if $R = R^{-1}$.

c) Let A be a set. Let R be the inclusion relation on on the set $\mathcal{P}(A)$. Prove that the partially ordered sets $(\mathcal{P}(A), R)$ and $(\mathcal{P}(A), R^{-1})$ are isomorphic.

Problem 2. Consider the set $A = \{2, 4, 6, 9, 18, 27, 36, 48, 60, 72\}$ ordered by divisibility.

a) Find all maximal and minimal elements.

b) Is there a largest or smallest element?

c) Find all upper bounds of $\{2, 9\}$. Is there a least upper bound?

d) Find all lower bounds of $\{48, 60, 72\}$. Is there a greatest lower bound?

e) Find all antichains with 4 elements.

f) Express A as a union of 4 disjoint chains. Can you do it with 3 chains? Explain your answer.

g) Find all predecessors and successors of 18.

Problem 3. Prove that if $(A, \leq_A)$ is a finite partially ordered set with n elements then there is a bijective order preserving function $f : A \longrightarrow \{1, 2, \ldots, n\}$ (where we use the usual ordering of $\{1, 2, \ldots, n\}$). Hint: use induction on n. The fact that A has a maximal element should be useful..

Problem 4. Let $(A, \leq)$ be a partially ordered set which contains 2 elements a, b which are not comparable but it does not contain any antichain with 3 elements.

a) Let X_a be the set of all elements which are not comparable with b and let X_b be the set of all elements not comparable with a. Prove that X_a and X_b are chains and that they are disjoint.

b) Let c be an element which is neither in X_a nor in X_b. Prove that c is either larger than both a and b or it is smaller than both a and b. Prove that either $X_a \cup \{c\}$ or $X_b \cup \{c\}$ is a chain.

c) Suppose in addition that A is finite. Prove that A can be expressed as a union of two disjoint chains (this is a bit harder problem).

Remark. A more general result, Dilworth's theorem, says that if a partially ordered set does not have an antichain with k elements then it can be expressed as a union of less than k pairwise disjoint chains.