**Homework 11**

due on Wednesday, August 3

Read carefully sections 1,2 of Chapter 2 in Shen's book and and sections 2.6, 2.7 in Stoll's book. Solve the following problems.

**Problem 1.** a) Prove that the sets $\mathbb{N} \times \mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ (with lexicographic order) are not isomorphis. Hint. Look at $(0,0)$ in $\mathbb{N} \times \mathbb{Z}$ and all the elements below it.

b) Consider the set $\mathbb{N} \times \mathbb{N}$ with the order $\leq_d$ definned as follows: $(a,b) \leq_d (c,d)$ if either $a+b < c+d$ or $a+b = c+d$ and $a \leq c$. Prove that $\leq_d$ is a linear order and that $(\mathbb{N} \times \mathbb{N}, \leq_d)$ is isomorphic to $\mathbb{N}$ with its natural order. Hint: Use a theorem from class which characterizes $\mathbb{N}$.

**Problem 2.** Let $(A, \leq)$ be a partially ordered set. Suppose that $a, b \in A$ are two elements which are not comparable. Define w new relation $\leq_n$ on $A$ as follows: $x \leq_n y$ iff either $x \leq y$ or $x \leq a$ and $b \leq y$. Prove that $\leq_n$ is a partial order on $A$ which extends $\leq$ and that $a \leq_n b$.

**Problem 3.** Let $(A, \leq)$ be a partially ordered set. Let $H(A)$ be the set of all antichains in $A$. In other words, elements of $H(A)$ are those subsets of $A$ which are antichains. Define a relation $\lll$ on $H(A)$ as follows: $X \lll Y$ if for any $x \in X$ there is $y \in Y$ such that $x \leq y$.

a) Prove that $\lll$ is a partial order on $H(A)$.

b) Let $X$, $Y$ be elements of $H(A)$. Let $Z$ be the set of all maximal elements in the set $X \cup Y$. Prove that $Z \in H(A)$ and that $Z$ is the least upper bound of $X$ and $Y$ with respect to the order $\lll$.

c) Suppose that $A$ has no atichains with $k+1$ elements. Let $X$ and $Y$ be antichains with $k$ elements. Let $Z$ be defined as in b) and let $W$ be the set of all minimal elements in the set $X \cup Y$. Prove that $Z$ and $W$ both have $k$ elements and that $W$ is in $H(A)$ and it is the largest lower bound for $X$ and $Y$.