Homework 13
due on Tuesday, August 9

Read carefully sections 6, 7, 8 of Chapter 2 in Shen's book and and sections 2.8, 2.9, 2.10, 2.11 in Stoll's book. Solve the following problems.

Problem 1. Let $(A, \leq_A)$ be a partially ordered set.

a) Consider a family of subsets of A, each of which is a chain in A. Suppose that for any two members X, Y in that family, either $X \subseteq Y$ or $Y \subseteq X$ (in other words, the family is a chain of subsets of A with respect to inclusion). Prove that the union of all members of that family is again a chain in A.

Start your proof as follows: Let U be the union of all members of the family and let a, b be in U. I need to prove that a and b are comparable. ...

b) Consider a family of subsets of A, each of which is an antichain in A. Suppose that for any two members X, Y in that family, either $X \subseteq Y$ or $Y \subseteq X$ (in other words, the family is a chain of subsets of A with respect to inclusion). Prove that the union of all members of that family is again an antichain in A.

Start your proof as follows: Let U be the union of all members of the family and let a, b be two distinct elements in U. I need to prove that a and b are not comparable. ...

c) Consider a family of subsets of A, each of which is an initial segment in A. Prove that both the union of all members of that family and the intesection of all members of that family are again initial segments in A.

Problem 2. Let $(A, \leq_A)$ be a partially ordered set. Consider the family $\mathcal{C}$ of all subsets of A which are chains in A. The set $\mathcal{C}$ is partially ordered by inclusion.

a) Prove that $\mathcal{C}$ is inductively ordered by inclusion (use the previous problem).

b) Use Kuratowski-Zorn Lemma to prove that there exist a chain X in A such that no element in $A \setminus X$ is comparable to all elements in X (i.e. no element can be added to X to form a bigger chain). This result is often called **Hausdorff's maximal principle**.

c) Assuming **Hausdorff's maximal principle** prove the Kuratowski-Zorn Lemma.

Problem 3. Let $(A, \leq_A)$ be a partially ordered set. Consider the family $\mathcal{N}$ of all subsets of A which are antichains in A. The set $\mathcal{N}$ is partially ordered by inclusion.

a) Prove that $\mathcal{N}$ is inductively ordered by inclusion (use Problem 1b).

b) Use Kuratowski-Zorn Lemma to prove that there exist an antichain Z in A such that every element in $A \setminus Z$ is comparable to some element in Z (i.e. no element can be added to Z to form a bigger antichain).