Homework 1

due on Thursday, July 14

Read carefully sections 1.8-1.9 in Stoll's book and the notes about functions linked on the course web page. Solve the following problems.

Problem 1. Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be functions.

a) Prove that if both f and g are surjective then gf is surjective.

b) Prove that if gf is injective then f is injective. Show by example that g need not be injective.

Problem 2. Let $f : A \longrightarrow B$ be a function.

a) Prove that if S, T are subsets of A then $f(S \cup T) = f(S) \cup f(T)$.

b) Prove that if S, T are subsets of B then $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$.

c) Prove that if S, T are subsets of B then $f^{-1}(S \bigtriangleup T) = f^{-1}(S) \bigtriangleup f^{-1}(T)$

d) Prove that if S, T are subsets of A then $f(S) \setminus f(T) \subseteq f(S \setminus T)$. Show by example that the equality does not always hold.

Remark. For any set A the power set $P(A)$ with operations $+$ (symmetric difference) and $\cdot$ (intersection) is a Boolean algebra. A function $f : A \longrightarrow B$ induces a function $f^{-1} : P(B) \longrightarrow P(A)$ and properties b), c) mean that f^{-1} is a homomorphism of boolean algebras. We also have induced function $P(A) \longrightarrow P(B)$ $(S \mapsto f(S))$, but usually it is not a homomorphism of Boolean algebras.

Problem 3. Let $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ be a function defined as follows: $f(m)$ is the last digit in the decimal expansion of m. What is the domain, codomain, range of f? Find $f(S)$, where S is the set of all even numbers.

b) What is $f \circ f$?