Homework 6

due on Thursday, July 20

Read carefully sections 4,6 of Chapter 1 in Shen's book and and sections 2.3-2.4 in Stroll's book. Solve the following problems.

Problem 1. a) Let $f : A \longrightarrow B$ be surjective. Let C be a subset of B. Prove that there is a surjective function from A onto C. (this is the same problem as in Hw 3. I want you to write carefully a correct solution this time)

b) Recall that $\mathcal{P}(X)$ is the set of all subsets of a set X. Define a function $h : \mathcal{P}(\mathbb{N}) \longrightarrow \mathbb{R}$ as follows: for a subset X of $\mathbb{N}$ set $h(X) = 0.a_0a_1a_2\ldots$, where $a_i = 1$ if $i \in X$ and $a_i = 0$ of $i \notin X$. Prove that h is injective.

c) Use a), b), and the fact that there is no surjective function from X onto $\mathcal{P}(X)$ to prove that there is no surjective function from $\mathbb{N}$ onto $\mathbb{R}$.

Problem 2. Let $\{A_i\}$, $i \in I$ be a collection of pairwise disjoint sets. Likewise, let $\{B_i\}$, $i \in I$ be a collection of pairwise disjoint sets. Suppose that $A_i \simeq B_i$ for every $i \in I$. Prove that $\bigcup_{i \in I} A_i \simeq \bigcup_{i \in I} B_i$.

Problem 3. Let A be a set. A function $F : \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ is called **increasing** if $F(T_1) \subseteq F(T_2)$ for any subsets T_1, T_2 of A such that $T_1 \subseteq T_2$ (recall that subsets of A are elements of $\mathcal{P}(A)$).

Let $F : \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ be an inreasing function. Consider the subset $\mathcal{Q} = \{T \in \mathcal{P}(A) : F(T) \subseteq T\}$ of $\mathcal{P}(A)$. Let S be the intesection of all sets which belong to $\mathcal{Q}$.

a) Prove that $A \in \mathcal{Q}$, so $\mathcal{Q}$ is not empty.

b) Let $T \in \mathcal{Q}$. Prove that $F(T) \in \mathcal{Q}$.

c) Let $T \in \mathcal{Q}$. Prove that $F(S) \subseteq T$. Conclude that $S \in \mathcal{Q}$. Thus S is the smallest element in $\mathcal{Q}$.

d) Use b) and c) to conclude that $F(S) = S$.

The moral of this problem is that every increasing function has a fixed point.