**Homework 7**

due on Friday, July 21

Read carefully sections 5 of Chapter 1 in Shen's book and and sections 2.3 in Stroll's book. Solve the following problems.

**Problem 1.** Let $f : A \longrightarrow B$ and $g : B \longrightarrow A$ be functions. Define a function $H : \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ as follows: $H(X) = A \setminus g(B \setminus f(X))$.

a) Prove that $H$ is an increasing function. Conclude that there is a subset $S$ of $A$ such that $H(S) = S$ (use Problem 3 from homework 6).

b) Prove that $g$ maps $B \setminus f(S)$ onto $A \setminus S$.

c) Suppose in addtion that both $f$ and $g$ are injective. Prove that $g$ gives a bijection form $B \setminus f(S)$ onto $A \setminus S$ and $f$ gives a bijection from $S$ onto $f(S)$.

d) Use c) to give another proof of the Cantor-Berenstein-Schroder Theorem. Hint: $A = (A \setminus S) \cup S$.

**Problem 2.** a) Let $A, B$ be sets. Prove that if $A \simeq B$ then $2^A \simeq 2^B$.

b) Let $A$ and $B$ be disjoint sets. Prove that $2^A \times 2^B \simeq 2^{A \cup B}$.

c) Prove that if $A$ is denumerable and $B$ is countable then $2^A \times 2^B \simeq \mathbb{R}$. Hint: Note first that one can assume that $A$ and $B$ are disjoint.

**Problem 3.** Let $A_0, A_1, ...$ be a sequence of sets such that $A_i \simeq \mathbb{R}$ for all $i$. Let $A = \bigcup_{i=0}^{\infty} A_i$ be the union of all these sets. Prove that $A \simeq \mathbb{R}$. Hint: Embed $A$ into $\mathbb{R} \times \mathbb{R}$, or even $\mathbb{R} \times \mathbb{N}$.