Homework 9
due on Thursday, July 27

Read carefully sections 1,2 of Chapter 2 in Shen's book and and sections 1.6, 1.7, 1.11 in Stroll's book. Solve the following problems.

Problem 1. Let R be a relation on a set A. Recall the following terminology.

R is **reflexive** if xRx for every $x \in A$.

R is **irreflexive** (or **antireflexive**) if xRx is false for every $x \in A$.

R is **symmetric** if, for any $x, y \in A$, xRy implies yRx.

R is **antisymmetric** if, for any $x, y \in A$, xRy and yRx imply $x = y$.

R is **asymetric** if, for any $x, y \in A$, xRy implies that yRx does not hold.

R is **transitive** if, for all $x, y, z \in A$, xRy and yRz imply xRz.

a) Let R be the relation $a|b$ (a divides b) on the set $\mathbb{N}$ of natural numbers (in other words, aRb means $a|b$). Which of the above properties hold for R. Carefully justify your answer for each property.

b) Let R_1 and R_2 be two equivalence relations on a set A (recall that this menas in particular that R_1 and R_2 are subsets of $A \times A$). Prove that $R_1 \cap R_2$ is also an equivalence relation.

Problem 2. Let $f : A \longrightarrow B$ be a function. Let R be a relation on A defined by $R = \{(x, y) \in A \times A : f(x) = f(y)\}$. In other words, xRy if and only if $f(x) = f(y)$. Prove that R is an equivalence relation. What are the equivalence classes of R? (describe them in terms of the notion of preimage).

Problem 3. Let $A = \{a, b, c\}$ be a set with three elements. Describe all possible equivalence relations on A (use the correspondence between eqivalence relations and partitions).

Problem 4. a) Let $\{A_i : i \in I\}$ be a collection of sets. Let $\leq_i$ be a partial order on A_i for each $i \in I$. Consider the product $P = \prod_{i\in I} A_i$. Recall that it consits of all functions $f : I \longrightarrow \bigcup_{i\in I} A_i$ such that $f(i) \in A_i$ for all $i \in I$. Define a relation $\leq$ on P as follows: $f \leq g$ if and only if $f(i) \leq_i g(i)$ for all $i \in I$. Prove that $\leq$ is a partial oredr on P. The partially ordered set $(P, \leq)$ is called the product of the partially ordered sets $(P_i, \leq_i)$, $i \in I$.

b) Let $(A_1, \leq_1)$, $(A_2, \leq_2)$,..., $(A_n, \leq_n)$ be partially ordered sets. Define a relation $\leq$ on $A = A_1 \times A_2 \times \ldots \times A_n$ as follows: $(a_1, a_2, \ldots, a_n) \leq (b_1, b_2, \ldots, b_n)$ if either $a_i = b_i$ for all i or for the smallest i such that $a_i \neq b_i$ we have $a_i \leq_i b_i$. Prove that $\leq$ is a partial order on A. Prove that if each order $\leq_1, \leq_2, \ldots, \leq_n$ is linear then $\leq$ is linear. The ordere $\leq$ is called the lexicographic order induced by the orders $\leq_1, \leq_2, \ldots, \leq_n$.