Solutions to Exam

Math 488A, 575A, 590A

Problem 1. a) What does it mean that sets A and B have the same cardinality? (5 points)

b) What does it mean that the cardinality of A is less than or equal to the cardinality of B? (5 points)

c) State the Cantor-Bernstein-Schröder Theorem. (5 points)

Solution: a) We say that the sets A, B have the same cardinality if there is a bijective function $f : A \longrightarrow B$.

b) We say that the cardinality of A is less than or equal to the cardinality of B if there is an injective function $f : A \longrightarrow B$.

c) Cantor-Bernstein-Schröder Theorem: Let A, B be sets such that there is an injective function from A into B and an injective function from B into A. Then there is a bijective function $f : A \longrightarrow B$.

Problem 2. a) What does it mean that a set is denumerable? What does it mean that a set is countable? (5 points)

b) Let A be a set with at least 2 elements. For $i = 1, 2, 3, \ldots$ let $f_i : \mathbb{N} \longrightarrow A$ be a function. Use Cantor's diagonal method to prove that there is a function $f : \mathbb{N} \longrightarrow A$ which is not equal to any of the functions $f_1, f_2, \ldots$. Conclude that the set $A^{\mathbb{N}}$ is not countable. (12 points)

Solution: a) A set A is called denumerable if A and $\mathbb{N}$ have the same cardinality, i.e. if there is a bijective function from $\mathbb{N}$ onto A. A set is countable if it is either finite or denumerable.

b) The idea is to define a function $f : \mathbb{N} \longrightarrow A$ which differs from f_1 at 1, differs from f_2 at 2, differs from f_3 at 3, etc. Choose two elements a, b in A (here we use the assumption that A has at least 2 elements). Define f as follows: given $k \in \mathbb{N}$ we define

$$f(k) = \begin{cases} a & \text{if } f_k(k) \neq a \\ b & \text{if } f_k(k) = a \end{cases}$$

Since, for every k, the functions f and f_k have different values at k, the function f is not equal to any of the functions $f_1, f_2, \ldots$.

Problem 3. a) Prove that $(A \setminus B) \cap (C \setminus D) = (A \cap C) \setminus (B \cup D)$ by starting with an element in the left hand side and proving that it is in the right hand side and vice versa. (6 points)

b) Use membership table to prove that $(A \setminus B) \triangle (A \setminus C) = A \cap (B \triangle C)$. (6 points)

c) Express each side of the equality

$$(A \setminus B) \cup (B \setminus C) = (A \cup B) \setminus (B \cap C)$$

using only the operation $+$ of symmetric difference and $\cdot$ of intersection. Then verify that both sides are indeed equal. (6 points)

Solution: a) Note that $x \in (A \setminus B) \cap (C \setminus D)$ if and only if $x \in (A \setminus B)$ and $x \in (C \setminus D)$, which is equivalent to the conditions $x \in A$ and $x \notin B$ and $x \in C$ and $x \notin D$, which in turn is equivalent to the conditions $x \in A$ and $x \in C$ and $x \notin B$ and $x \notin D$ which is equivalent to $x \in (A \cap C)$ and $x \notin (B \cup D)$,

which is equivalent to $x \in (A \cap C) \setminus (B \cup D)$. Thus the sets $(A \setminus B) \cap (C \setminus D)$ and $(A \cap C) \setminus (B \cup D)$ have the same elements and therefore they are equal.
Remark. To be on a safe side, one can do the two inclusions $(A \setminus B) \cup (B \setminus C) \subseteq (A \cup B) \setminus (B \cap C)$ and $(A \cup B) \setminus (B \cap C) \subseteq (A \setminus B) \cup (B \setminus C)$ separately.

b)

A	B	C	$A \setminus B$	$A \setminus C$	$(A \setminus B) \vartriangle (A \setminus C)$	$B \vartriangle C$	$A \cap (B \vartriangle C)$
1	1	1	0	0	0	0	0
1	1	0	0	1	1	1	1
1	0	1	1	0	1	1	1
1	0	0	1	1	0	0	0
0	1	1	0	0	0	0	0
0	1	0	0	0	0	1	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Since the columns for $(A \setminus B) \vartriangle (A \setminus C)$ and $A \cap (B \vartriangle C)$ are equal, we conclude that $(A \setminus B) \vartriangle (A \setminus C) = A \cap (B \vartriangle C)$.
Remark. The first three columns of the table describe all possible membership patterns. For example, the first row corresponds to elements which belong to all three sets A, B, C, and the 4th row corresponds to elements which belong to A but do not belong to B or C. The second part of the table is then filled by going through each row and putting 1 if the elements described by the first part of the row belong to the set naming a given column and putting 0 if they do not belong to this set. For example, in the second row in the column for $A \setminus C$ we put 1, since elements in this row belong to A, B but not C so they belong to $A \setminus C$.

c) Recall that $X \setminus Y = X + XY$ and $X \cup Y = X + Y + XY$. We have

$$(A \setminus B) \cup (B \setminus C) = (A+AB)+(B+BC)+(A+AB)(B+BC) = A+AB+B+BC+AB+ABC+ABB+$$
$$+ABBC = A+B+AB+BC+AB+AB+ABC+ABC = A+B+AB+BC.$$

(we use the properties $X + X = 0$ and $XX = X$). Similarly,

$$(A \cup B) \setminus (B \cap C) = (A+B+AB)+(A+B+AB)BC = A+B+AB+ABC+BBC+ABBC =$$
$$= A+B+AB+BC+ABC+ABC = A+B+AB+BC.$$

We see that both $(A \setminus B) \cup (B \setminus C)$ and $(A \cup B) \setminus (B \cap C)$ are equal to $A+B+AB+BC$, hence they are equal to each other.

Problem 4. a) Let A be a subset of B. Prove that for any set C we have $A^C \preccurlyeq B^C$. (6 points)

b) Using a) and the arithmetic of cardinal numbers prove that $\mathbb{N}^{\mathbb{N}} \simeq \mathbb{R}$. State all results you are using. (10 points)

Solution: a) Since A is a subset of B, any function which maps C into A is automatically a function from C into B. This shows that $A^C \subseteq B^C$ and therefore $A^C \preccurlyeq B^C$.

We will prove a more general result, that if $A \preccurlyeq B$ then $A^C \preccurlyeq B^C$. Indeed, since $A \preccurlyeq B$, there is an injective function $h : A \longrightarrow B$. Using this function, we define a function $\Phi : A^C \longrightarrow B^C$ as follows: for any $f \in A^C$, i.e. $f : C \longrightarrow A$ we set $\Phi(f) = hf$. Clearly hf is a function from C into B, so $\Phi(f) \in B^C$. We need to prove that Φ is injective. Suppose that $\Phi(f_1) = \Phi(f_2)$. Then $hf_1 = hf_2$.

Thus, for any $c \in C$ we have $h(f_1(c)) = h(f_2(c))$. Since h is injective, we conclude that $f_1(c) = f_2(c)$. Since c was an arbitrary element of C, we see that $f_1 = f_2$. This proves that $\Phi : A^C \longrightarrow B^C$ is injective. It follows that $A^C \preccurlyeq B^C$. In the language of cardinal numbers this result means that if α, β, γ are cardinal numbers and $\alpha \leq \beta$ then $\alpha^\gamma \leq \beta^\gamma$.

b) Let $\aleph_0 = |\mathbb{N}|$. Then, as we proved, $|\mathbb{R}| = 2^{\aleph_0}$. Since $2 \leq \aleph_0$, we get from part a) that $2^{\aleph_0} \leq \aleph_0^{\aleph_0}$. On the other hand, $\aleph_0 \leq 2^{\aleph_0}$, so $\aleph_0^{\aleph_0} \leq (2^{\aleph_0})^{\aleph_0}$ (again by part a)). But $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}$, as $\aleph_0 \cdot \aleph_0 = \aleph_0$. Thus we see that $2^{\aleph_0} \leq \aleph_0^{\aleph_0}$ and $\aleph_0^{\aleph_0} \leq 2^{\aleph_0}$. By the Cantor-Bernstein-Schröder theorem, we get $\aleph_0^{\aleph_0} = 2^{\aleph_0}$. This means that $\mathbb{N}^{\mathbb{N}} \simeq \mathbb{R}$, as $|\mathbb{N}^{\mathbb{N}}| = \aleph_0^{\aleph_0}$ and $|\mathbb{R}| = 2^{\aleph_0}$.

Problem 5. a) Let $f : A \longrightarrow B$ be a surjective functions. Let $g : B \longrightarrow C$, $h : B \longrightarrow C$ be functions such that $gf = hf$. Prove that $g = h$. Show by example that the result is no longer true when the assumption that f is surjective is dropped. (8 points)

b) Let $f : X \longrightarrow Y$ be a function and let A, B be subsets of Y. Prove that
$f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$. Start your solution with "Let $u \in f^{-1}(A \setminus B)$". (8 points)

Solution: a) Let $b \in B$. We need to prove that $g(b) = h(b)$. Since f is surjective, there is $a \in A$ such that $f(a) = b$. Then $g(b) = g(f(a)) = (gf)(a) = (hf)(a) = h(f(a)) = h(b)$. Since b is an arbitrary element of B, we get $g = h$.

b) Let $x \in f^{-1}(A \setminus B)$. This means that $f(x) \in A \setminus B$, i.e. $f(x) \in A$ and $f(x) \notin B$. Thus $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$, which implies that $x \in f^{-1}(A) \setminus f^{-1}(B)$. This proves that $f^{-1}(A \setminus B) \subseteq f^{-1}(A) \setminus f^{-1}(B)$.

Conversely, suppose that $x \in f^{-1}(A) \setminus f^{-1}(B)$. Then $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$, so $f(x) \in A$ and $f(x) \notin B$. Thus $f(x) \in A \setminus B$ and therefore $x \in f^{-1}(A \setminus B)$. This proves that $f^{-1}(A) \setminus f^{-1}(B) \subseteq f^{-1}(A \setminus B)$.

Since we proved that $f^{-1}(A \setminus B) \subseteq f^{-1}(A) \setminus f^{-1}(B)$ and $f^{-1}(A) \setminus f^{-1}(B) \subseteq f^{-1}(A \setminus B)$, we conclude that $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.

Problem 6. a) State the definition of a relation R on a set A. What does it mean that R is transitive? What does it mean that R is antisymmetric? Define a relation R on the set $A = \{1, 2, 3, 4\}$ which is reflexive, neither symmetric nor antisymmetric. (8 points)

b) Let $\leq$ be a partial order on a set A. Suppose that a, b are two elements of A which are not comparable. Let A_a be the set of all elements comparable with a. Similarly, let A_b be the set of all elements comparable with b. Prove that if $x \in A_a \cap A_b$ then either x is smaller than each of a, b or x is larger that each of a, b. (5 points)

c) Let A be a set. Consider the following relation R on the set $\mathcal{P}(A)$: XRY if and only if $X \triangle Y$ is countable. Prove that R is an equivalence relation. What is the equivalence class of the empty set? Hint: $(X \triangle Y) \triangle (Y \triangle Z) = X \triangle Z$ (5 points).

Solution: a) A **relation** R on a set A is a subset R of $A \times A$. Instead of writing $(a, b) \in R$ one often writes aRb and says that a is in relation R with b.
R is **transitive** if for any $a, b, c \in A$ such that aRb and bRc, we have aRc.
R is **antisymmetric** if for any $a, b \in A$, if aRb and bRb,then $a = b$. In other words, if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$.

We define the relation R by listing all its elements. Since R is supposed to be reflexive, it must contain $(1, 1), (2, 2), (3, 3), (4, 4)$. To make R not symmetric we will add $(1, 2)$ but not $(2, 1)$. To ensure that it is not antisymmetric we add $(3, 4)$ and $(4, 3)$. Thus the relation

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (3, 4), (4, 3)\}$$

has all the required properties.

b) If $x \in A_a \cap A_b$ then x is comparable with both a and b and x is not equal to any of a, b. We have therefore four possibilities:

$a < x$ and $x < b$
$x < a$ and $b < x$
$a < x$ and $b < x$
$x < a$ and $x < b$.

In the first case, we get $a < b$ by transitivity, which contradicts the assumption that a and b are not comparable. Similarly, second case yields $b < a$, a again a contradiction. Thus only the last two cases are possible, i.e. either x is smaller than each of a, b or x is larger that each of a, b.

c) We need to prove that the relation R is reflexive, symmetric, and transitive.

Since for any set X we have $X \bigtriangleup X = \emptyset$ and the empty set is countable, we see that XRX, i.e. R is reflexive.

Suppose now that XRY. Then $X \bigtriangleup Y$ is countable. But symmetric difference is commutative, os $Y \bigtriangleup X$ is countable, i.e. YRX. This proves that R is symmetric.

Suppose that XRY and YRZ. Then $X \bigtriangleup Y$ and $Y \bigtriangleup Z$ are countable sets. We have $X \bigtriangleup Z = (X \bigtriangleup Y) \bigtriangleup (Y \bigtriangleup Z)$. Now if B, C are countable sets then so are $B \setminus C$ and $C \setminus B$ (since a subset of a countable set is countable). It follows that $B \bigtriangleup C = (B \setminus C) \cup (C \setminus B)$ is also countable, since union of two countable sets is countable. This shows that the symmetric difference of two countable sets is countable. It follows that $X \bigtriangleup Z$ is countable, i.e. XRZ. This proves that R is transitive.

The equivalence class of $\emptyset$ consists of all subsets X of A such that $\emptyset RX$. This means that $\emptyset \bigtriangleup X = X$ is countable. Thus the equivalence class of $\emptyset$ consists of all countable subsets of A.