Expected Value and Variance of a Bernoulli trial

If $Y = Y(\omega)$ is a "Bernoulli variable" with success probability $P(Y = 1) = p$ and failure probability $P(Y = 0) = q = 1 - p$ then

(a) $E(Y) = p$,

(b) $\mathrm{Var}(Y) = pq$.

We show **(a)** although you saw in lecture that $E(Y) = p$. So here it is again:

$$E(Y) \;=\; \sum_y y \cdot P(Y = y) \;=\; 1 \cdot P(Y = 1) + 0 \cdot P(Y = 0) \;=\; 1 \cdot p + 0 \cdot q \;=\; p.$$

We show **(b):** Seeing that $\mathrm{Var}(Y) = pq$ needs a lot more arithmetic.

$$\mathrm{Var}(Y) \;=\; \sum_y \left(y - E(Y)\right)^2 \cdot P(Y = y) \;=\; \sum_y \left(y - E(Y)\right)^2 \cdot P(Y = y) \quad \text{since } E(Y) = p.$$

That is again a sum of two terms since the only possible outcomes are $y = 1$ (success) and $y = 0$ (failure). Thus

$$\mathrm{Var}(Y) \;=\; (1 - p)^2 \cdot p + (0 - p)^2 \cdot q \;=\; (1 - 2p + p^2)p + (-p)^2 \cdot (1 - p),$$
$$= p - 2p^2 + p^3 + p^2 - p^3 \;=\; p - p^2 \;=\; p(1 - p) \;=\; pq.$$