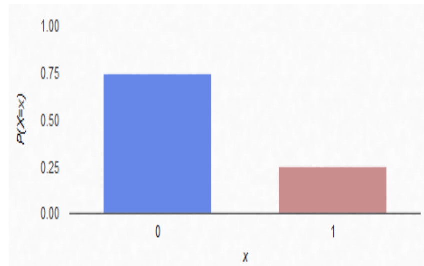
Binomial and Geometric Models

A **binary random phenomenon** is a random phenomenon (need not be numeric) X which has exactly two outcomes:

- "success" with probability p,
- "failure" with probability $q = 1 - p$.

So X has a probablity distribution as seen to the right.
(In this example: $p = 0.25$, success = 1, failure = 0)



Given is a finite or infinite sequence $X_1, X_2, X_3, \ldots$ of random phenomena (need not be binary).

a. If each X_j has the same distribution (the process does not change with j: we get the same probability histogram for each j), we say that
- $X_1, X_2, X_3, \ldots$ are **identically distributed**.

b. If, MOREOVER, each trial X_j is independent of the others, we say that
- $X_1, X_2, X_3, \ldots$ are "**iid**": **independent and identically distributed**.

If we have a finite or infinite sequence $X_1, X_2, X_3, \ldots$ of binary random phenomena then we call each one of those X_j a **Bernoulli trial** rather than a binary random phenomenon if the sequence is iid.

In other words,

$X_1, X_2, X_3, \ldots$ are Bernoulli trials if

a. each X_j has exactly two outcomes: success or failure,

b. the success probability $p = P(X_j) =$ "success" does not change with j (hence each X_j has the same probability distribution),

c. the X_j are independent.

Examples of Bernoulli trials:

a. X_j = j–th toss of a fair coin: success = Heads; $p = 0.5, q = 0.5$

b. X_j = j–th throw of a fair die: success = 5 or 6; $p = 2/6 = 1/3 = 0.333, q = 2/3$.

c. X_j = j–th opening of a cereal box: success = Hope Solo picture; $p = 0.2, q = 0.8$

Convert to (numerical) random variables: Change success to 1 and failure to 0.

a. X_j = j–th toss of a coin: assign 1 if Heads, assign 0 if Tails; $p = 0.5$

b. X_j = j–th throw of a die: assign 1 if 5 or 6; assign 0 otherwise; $p = 2/6 = 0.333$.

c. X_j = j–th opening of a cereal box: assign 1 if Hope Solo picture; assign 0 otherwise; $p = 0.2$

Write s for success, f for failure

Geometric Model GEOM(p):

- Sequence of (iid) Bernouli trials $X_1, X_2, X_3, \ldots$ with success probability $p = P(X_j = s)$
- T := first index n such that X_n = success (random "time"!)
- Outcome $\{T = n\}$ same as $\{X_1 = f$ **and** $X_2 = f$ **and** $\ldots$ **and** $X_{n-1} = f$ **and** $X_n = s\}$.
- Product rule (the X_j are independent!)):
  $P(T = n) \ = \ P(X_1 = f) \times P(X_2 = f) \times \cdot \times P(X_{n-1} = f) \times P(X_n = s)$
- $= q \cdot q \cdots q \cdot p \ = \ q^{i-1} \cdot p$

Example: Repeatedly rolling a die.

- Compute P(first 1 comes at the 4–th throw).
- Solution: X_j = j–th roll; $p = P(X_j = 1) = 1/6$; $P(T = 4) = q \cdot q \cdot q \cdot p$
  $= (5/6)^3 \cdot 1/6 = (125/216)/6 \approx 0.965 = 9.65\%$

Binomial Model BINOM(n,p):

- Finite sequence of n (iid) Bernouli trials $X_1, X_2, X_3, \ldots X_n$
  with success probability $p = P(X_j = s)$,
- Encode success as 1, failure as 0, so $p = P(X_j = 1)$,
- $S_n := X_1 + X_2 + \cdots + X_n$ = # of successes in those n trials
- Probability of k successes in n trials is

$$P(S_n = k) \ = \ \binom{n}{k} \cdot p^k \cdot q^{n-k}; \ \text{Binomial coefficient} \binom{n}{k} \ = \ {}_nC_k \ = \ \frac{n!}{k!(n-k)!}$$

Example: Toss a coin 6 times ($n = 6$)

- Compute P(exactly 2 tails).
- Solution: X_j = j–th toss; $p = P(\text{Tails}) = P(X_j = 1) = 1/2$;
- P(exactly 2 tails) = $P(S_6 = 2) = \binom{6}{2} \cdot (1/2)^2 \cdot (1 - 1/2)^{6-2}$
  $= \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1)(2 \cdot 1)} \cdot 0.25 \cdot 0.0625 = \frac{6 \cdot 5}{2 \cdot 1} \cdot 0.25 \cdot 0.0625 \approx 0.23 = 23\%$