Math 330 - Additional Material

Student edition with proofs

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Contents

1 Before You Start

Errors detected by Math 330 students, Spring 2017:

Date	Topic
2017-01-26	*Error in def.4.2.* ***Incorrect version:*** *A relation is symmetric if $x_1 R x_2$ implies $x_1 R x_2$ for all $x_1, x_2 \in X$.* ***Correct version:*** *A relation is symmetric if $x_1 R x_2$ implies $x_2 R x_1$ for all $x_1, x_2 \in X$. Detected by* ***Brad Whistance****.*

History of Updates:

Date	Topic
2017-01-30	*Mainly reformatting: increased use of tables. Significant additions to ch.**??** (Appendix: Addenda to Beck/Geoghegan's "The Art of Proof").*
2017-01-10	*Many updates during Fall 2016. Significant streamlining and reorg of ch.8 (liminf, limsup, ...) through ch.13 (Zorn's Lemma).*

1.1 About This Document

Remark 1.1 (The purpose of this document)**.** This write-up was originally written in 2005 under the title "Introduction to Abstract Math – A Journey to Approximation Theory" and parts of it now serve as lecture notes for the course "Math 330: Number systems" which is held at the Department of Mathematical Sciences at Binghamton University.

These notes serve at least two purposes:

a. They provide background material on topics that cannot found in sufficient detail in [1] B/G (Beck/Geoghegan): The Art of Proof. This document will often be simply referred to as "B/G". It serves as the primary reference for the first two thirds of the Math 330 course.

b. They cover material which is beyond the scope of [1] B/G such as

- material on lim inf and lim sup
- convergence, continuity and compactness in metric spaces
- applications of Zorn's lemma

These topics are usually covered in the last third of my Math 330 class. □

Remark 1.2 (Acknowledgements)**.** The early chapters of this document draw on the following chapters of [4] Bryant, Kirby Course Notes for MAD 2104:

Ch.1, section 1: Introduction to Sets
Ch.1, section 2: Introduction to Functions
Ch.2: Logic
Ch.3: Methods of Proofs
Ch.4, section 1: Set Operations
Ch.4, section 2: Properties of Functions

Moreover such a document cannot be written with the intent to supplement the [1] B/G book without strongly borrowing from it. □

Remark 1.3 (How to navigate this document I)**.** Scrutinize the table of contents, including the headings for the subchapters. You will find many entries there tagged with a directive.

For example, the reference to ch.8 (The Completeness of the Real Numbers System) has been tagged with **(Understand this!)**,
the first subchapter ch.8.1 (Minima, Maxima, Infima and Suprema) has been tagged with **(Study this!)**,
the subchapter ch.8.4 (Sequences of Sets and Indicator functions and their liminf and limsup) has no tag.

Notation Alert:

All directives apply to the entire subtree and a lower level directive overrides the "parent directives". Example: the "Understand this!" directive of subsection 10.2.3: Continuity of Polynomials overrides the "Study this!" directive of subsection 10.2 on Continuity. Accordingly, when you do not see any comment, back up in the table of contents: first to the parent entry, then to its parent entry . . . until you find one.

a. "Study this" directive: When you read "Study this", you must understand the material in depth. You will need to do this with paper and pencil in hand and make an effort not only to understand what the definitions and theorems are all about – not a minor undertaking because some of the subject matter is quite abstract – but aso make an effort to follow the proofs at least from a birds eye perspective.

b. "Understand this" directive: When you read "Understand this", you should know the definitions, propositions and theorems without worrying about proofs. Chances are that the material will be referred to from truly important sections of this write-up and is primarily needed for their understanding.

b. "Skim this" directive: When you read "Skim this", you should understand how the material is structured. You may find it easier to do some of your homework. A good example is chapter 3 on logic. It has been marked as "Skim this" but some of the later subchapters override this as "Understand this" and you will have problems doing so unless you can find your way around in the material that precedes them.

b. "Skip this" directive: When you read "Skip this", you need not worry about the content.

You will find almost every week reading assignments as part of your homework. The reading is due prior to when it is needed in class, both for this document and the Beck/Geoghegan text. I assume that you did your reading and I will assume in particular that you have learned the definitions so that I can move along at a fast pace except for some topics that I will focus on in detail. For this document it means that you should do the "Study this" and "Understand this" material as I indicated above. □

Remark 1.4 (How to navigate this document II)**.** My theory is that, particularly in Math, more words take a lot less time to understand than a skeletal write-up like the one given in the course text. Accordingly, almost all of the "Study this" and "Understand this"material provided in this document comes with quite detailed proofs. Those proofs are there for you to study.

Some of those proofs, notably those in prop. 6.2, make use of "⇔" to show that two sets are equal. You should study this technique but, as you will hear me say many times in class, I recommend that you abstain from using "⇔" between statements in your proofs. Chances are that you very likely lack the experience to do so without error.

Some of the material was written from scratch, other material was pulled in from a document that was written as early as 10 years ago. I have make an attempt to make the entire document more homogeneous but there will be some inconsistencies. Your help in pointing out to me the most notable trouble spots would be deeply appreciated.

There are differences in style: the original document was written in a much more colloquial style as it was addressed to talented high school students who had expressed a special interest in studying college level math. □

This is a living document: material will be added as I find the time to do so. Be sure to check the latest PDF frequently. You certainly should do so when an announcement was made that this document contains new additions and/or corrections.

1.2 How to Properly Write a Proof (Study this!)

Study this brief chapter to understand some of the dos and don'ts when submitting your homework.

To prove an equation such as $A = Z$ you are asked to do one of the following:

a.

$$\begin{aligned} A &= B \quad \textit{(use ....)} \\ &= C \quad \textit{(use ....)} \\ &= D \quad \textit{(use ....)} \\ &\dots\dots\dots\dots \\ &= Z \quad \textit{(use ....)} \end{aligned}$$

You then conclude from the transitivity of equality that $A = Z$ is indeed true.

b. *You transform the left side (L.S.) and the right side (R.S.) separately and show that in each case you obtain the same item, say M:*

Left side:

$$\begin{aligned} A &= B \quad \textit{(use ....)} \\ &= C \quad \textit{(use ....)} \\ &= D \quad \textit{(use ....)} \\ &\dots\dots\dots\dots \\ &= M \quad \textit{(use ....)} \end{aligned}$$

Right side:

$$\begin{aligned} Z &= Y \quad \textit{(use ....)} \\ &= X \quad \textit{(use ....)} \\ &= W \quad \textit{(use ....)} \\ &\ldots\ldots\ldots\ldots \\ &= M \quad \textit{(use ....)} \end{aligned}$$

and you rightfully conclude that the proof is done because it follows from $A = M$ and $Z = M$ that $A = Z$.

c. *Instead you may choose to proceed as follows*

$$\begin{aligned} A &= Z \quad \textit{(that's what you want to prove)} \\ B &= Y \quad \textit{(you do with both A and Z the same operation ......)} \\ C &= X \quad \textit{(you do with both B and Y the same operation ......)} \\ D &= W \quad \textit{(you do with both C and X the same operation ......)} \\ &\ldots\ldots\ldots\ldots \\ M &= M \quad \textit{(you do with both L and N the same operation ......)} \end{aligned}$$

What is potentially wrong with that last approach?

*In the abstract the issue is that when using method **a** or **b** you take in each step an equation that is known to be true or that you assume to be true and you rightfully conclude by the use of transitivity that you have proved what you wanted to be true.*

When you use method c you take an equation that you want to be true ($A = Z$) but have not yet proved to be so. If you are wrong then doing the same thing to both sides may potentially lead to two things that are equal.

*Here is a simple example that demonstrates why **method c is not allowed** for a mathematical proof. This method will be used in two different ways to prove that $-2 = 2$.*

First proof:

$$\begin{aligned} -2 &= 2 \quad \textit{(want to prove)} \\ -2 \cdot 0 &= 2 \cdot 0 \quad \textit{(multiply both sides from the right w. 0)} \\ 0 &= 0 \quad \textit{(B/G ax.1.2 (additive neutral element)} \end{aligned}$$

We are done. ■

Second proof:

$$\begin{aligned} -2 &= 2 \quad \textit{(want to prove)} \\ (-2)^2 &= 2^2 \quad \textit{(square both sides)} \\ 4 &= 4 \quad \textit{(obvious)} \end{aligned}$$

We are done. ■

*Now you know why you must never use method **c**.* [1]

[1] You will learn later in this chapter about injective functions which guarantee that if you do an operation (apply a function) to two different items then the results will also be different. If method c was restricted to only such operations then there would not be a problem. In the two "proofs" that show $-2 = 2$ we use operations that are not injective: In the first proof the assignment $x \mapsto 0 \cdot x$ throws everything into the same result zero. The second proof employs the assignment $x \mapsto x^2$ which maps two numbers x, y that differ by sign only to the same squared value $x^2 = y^2$.

2 Preliminaries about Sets, Numbers and Functions (Understand this!)

2.1 Sets and Basic Set Operations

Ask a mathematician how her or his Math is different from the kind of Math you learn in high school, in fact, from any kind of Math you find outside textbooks for mathematicians and theoretical physicists. One of the answers you are likely to get is that Math is not so much about numbers but also about other objects, among them sets and functions. Once you know about those, you can tackle sets of functions, set functions, sets of set functions, ...

An entire book can be filled with a mathematically precise theory of sets. [2] *For our purposes the following "naive" definition suffices:*

Definition 2.1 (Sets)**.** A **set** is a collection of stuff called **members** or **elements** which satisfies the following rules: The order in which you write the elements does not matter and if you list an element two or more times then **it only counts once**.

We write a set by enclosing within curly braces the elements of the set. This can be done by listing all those elements or giving instructions that describe those elements. For example, to denote by X the set of all integer numbers between 18 and 24 we can write either of the following:

$$X := \{18, 19, 20, 21, 22, 23, 24\} \quad \text{or} \quad X := \{n : n \text{ is an integer and } 18 \leqq n \leqq 24\}$$

Both formulas clearly define the same collection of all integers between 18 and 24. On the left the elements of X are given by a complete list, on the right **setbuilder notation**, i.e., instructions that specify what belongs to the set, is used instead.

It is customary to denote sets by capital letters and their elements by small letters but this is not a hard and fast rule. You will see many exceptions to this rule in this document.

We write $x_1 \in X$ to denote that an item x_1 is an element of the set X and $x_2 \notin X$ to denote that an item x_2 is not an element of the set X

For the above example we have $20 \in X$, $27 - 6 \in X$, $38 \notin X$, 'Jimmy' $\notin X$. □

Example 2.1 (No duplicates in sets)**.** The following collection of alphabetic letters is a set:

$$S_1 = \{a, e, i, o, u\}$$

and so is this one:

$$S_2 = \{a, e, e, i, i, i, o, o, o, o, u, u, u, u, u\}$$

Did you notice that those two sets are equal? □

Remark 2.1. The symbol n in the definition of $X = \{n : n \text{ is an integer and } 18 \leqq n \leqq 24\}$ is a **dummy variable** in the sense that it does not matter what symbol you use. The following sets all are equal to X:

$$\{x : x \text{ is an integer and } 18 \leqq x \leqq 24\},$$
$$\{\alpha : \alpha \text{ is an integer and } 18 \leqq \alpha \leqq 24\},$$
$$\{\mathfrak{z} : \mathfrak{z} \text{ is an integer and } 18 \leqq \mathfrak{z} \leqq 24\}$$

[2] See remark 2.2 ("Russell's Antinomy") below.

□

Remark 2.2 (Russell's Antinomy)**.** Care must be taken so that, if you define a set with the use of setbuilder notation, no inconsistencies occur. Here is an example of a definition of a set that leads to contradictions.

$$A := \{B : B \text{ is a set and } B \notin B\} \tag{2.1}$$

What is wrong with this definition? To answer this question let us find out whether or not this set A is a member of A. Assume that A belongs to A. The condition to the right of the colon states that $A \notin A$ is required for membership in A, so our assumption $A \in A$ must be wrong. In other words, we have established "by contradiction" that $A \notin A$ is true. But this is not the end of it: Now that we know that $A \notin A$ it follows that $A \in A$ because A contains **all** sets that do not contain themselves.

In other words, we have proved the impossible: both $A \in A$ and $A \notin A$ are true! There is no way out of this logical impossibility other than excluding definitions for sets such as the one given above. It is very important for mathematicians that their theories do not lead to such inconsistencies. Therefore, examples as the one above have spawned very complicated theories about "good sets". It is possible for a mathematician to specialize in the field of axiomatic set theory (actually, there are several set theories) which endeavors to show that the sets are of any relevance in mathematical theories do not lead to any logical contradictions.

The great majority of mathematicians take the "naive" approach to sets which is not to worry about accidentally defining sets that lead to contradictions and we will take that point of view in this document. □

Definition 2.2 (empty set)**.** $\emptyset$ or {} denotes the **empty set**. It is the one set that does not contain any elements. □

Remark 2.3 (Elements of the empty set and their properties)**.** You can state anything you like about the elements of the empty sets as there are none. The following statements all are true:

a: If $x \in \emptyset$ then x is a positive number.
b: If $x \in \emptyset$ then x is a negative number.
c: Define $a \sim b$ if and only if both are integers and $a - b$ is an even number. For any $x, y, z \in \emptyset$ it is true that
  c1: $x \sim x$,
  c2: if $x \sim y$ then $y \sim x$,
  c3: if $x \sim y$ and $y \sim z$ then $x \sim z$.
d: Let A be any set. If $x \in \emptyset$ then $x \in A$.

As you will learn later, **c:** means that "$\sim$" is an equivalence relation (see def.4.3 on p.70) and **d:** means that the empty set is a subset (see the next definition) of any other set. □

Definition 2.3 (subsets and supersets)**.** We say that a set A is a **subset** of the set B and we write $A \subseteq B$ if any element of A also belongs to B. Equivalently we say that B is a **superset** of the set A and we write $B \supseteq A$. We also say that B includes A or A is included by B. Note that $A \subseteq A$ and $\emptyset \subseteq A$ is true for any set A.

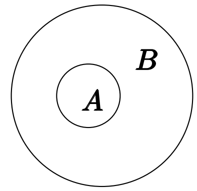

Figure 2.1: Set inclusion: $A \subseteq B, \; B \supseteq A$

If $A \subseteq B$ but $A \neq B$, i.e., there is at least one $x \in B$ such that $x \notin A$, then we say that A is a **strict subset** of B. We write "$A \subsetneq B$" or "$A \subset B$". Alternatively we say that B is a **strict superset** of A and we write "$B \supsetneq A$") or "$B \supset A$". $\square$

Two sets A and B are equal means that they both contain the same elements. In other words, $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

*"**iff**" is a short for "if and only if": P iff Q for two statements P and Q means that if P is valid then Q is valid and vice versa.* [3]

To show that two sets A and B are equal you show that
***a.** if $x \in A$ then $x \in B$,*
***b.** if $x \in B$ then $x \in A$.*

Definition 2.4 (unions, intersections and disjoint unions)**.** Given are two arbitrary sets A and B. No assumption is made that either one is contained in the other or that either one contains any elements!

The **union** $A \cup B$ (pronounced "A union B") is defined as the set of all elements which belong to A or B or both.

The **intersection** $A \cap B$ (pronounced "A intersection B") is defined as the set of all elements which belong to both A and B.

We call A and B **disjoint** if $A \cap B = \emptyset$. We then usually write $A \uplus B$ (pronounced "A disjoint union B") rather than $A \cup B$. $\square$

Definition 2.5 (set differences and symmetric differences)**.** Given are two arbitrary sets A and B. No assumption is made that either one is contained in the other or that either one contains any elements!

The **difference set** or **set difference** $A \setminus B$ (pronounced "A minus B") is defined as the set of all elements which belong to A but not to B:

$$A \setminus B := \{x \in A : x \notin B\} \tag{2.2}$$

[3] A formal definition of "if and only if" will be given in def.3.12 on p.34 where we will also introduce the symbolic notation $P \Leftrightarrow Q$. Informally speaking, a statement is something that is either true or false.

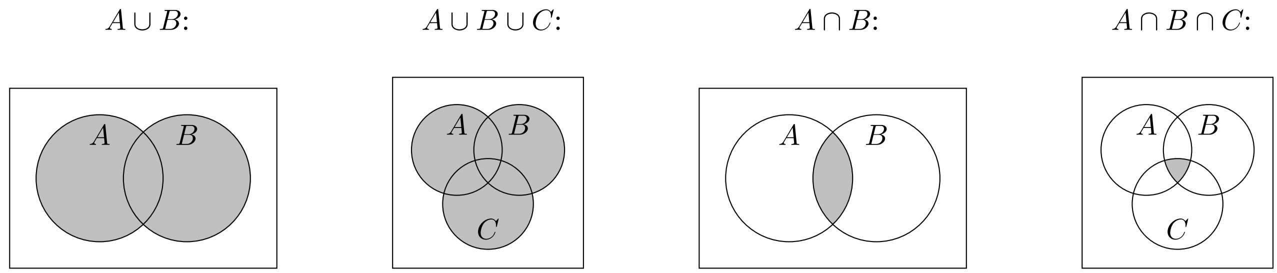

Figure 2.2: Union and intersection of sets

The **symmetric difference** $A \triangle B$ (pronounced "A delta B") is defined as the set of all elements which belong to either A or B but not to both A and B:

(2.3) $$A \triangle B := (A \cup B) \ \setminus \ (A \cap B) \ \square$$

Definition 2.6 (Universal set)**.** Usually there always is a big set Ω that contains everything we are interested in and we then deal with all kinds of subsets $A \subseteq \Omega$. Such a set is called a **“universal” set**. $\square$

For example, in this document, we often deal with real numbers and our universal set will then be $\mathbb{R}$.

If there is a universal set, it makes perfect sense to talk about the complement of a set:

Definition 2.7 (Complement of a set)**.** The **complement** of a set A consists of all elements of Ω which do not belong to A. We write $A^{\complement}$. or $\complement A$ In other words:

(2.4) $$A^{\complement} := \complement A := \Omega \setminus A = \{\omega \in \Omega : x \notin A\} \ \square$$

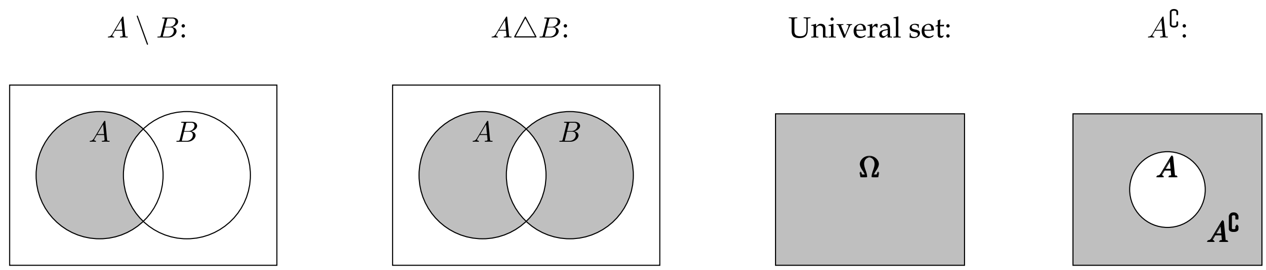

Figure 2.3: Difference, symmetric difference, universal set, complement

Remark 2.4 (Complement of empty, all)**.** Note that for any kind of universal set Ω it is true that

(2.5) $$\Omega^{\complement} = \emptyset, \qquad \emptyset^{\complement} = \Omega. \ \square$$

Example 2.2 (Complement of a set relative to the unit interval)**.** Assume we are exclusively dealing with the unit interval, i.e., $\Omega = [0,1] = \{x \in \mathbb{R} : 0 \leqq x \leqq 1\}$. [4] Let $a \in [0,1]$ and $\delta > 0$ and

$$A = \{x \in [0,1] : a - \delta < x < a + \delta\} \tag{2.6}$$

the δ–neighborhood [5] of a (with respect to $[0,1]$ because numbers outside the unit interval are not considered part of our universe). Then the complement of A is

$$A^{\complement} = \{x \in [0,1] : x \leqq a - \delta \text{ or } x \geqq a + \delta\}. \quad \square$$

Draw some Venn diagrams to visualize the following formulas.

Proposition 2.1. *Let A, B, X be sets and assume $A \subseteq X$. Then*

$$A \cup \emptyset = A; \qquad A \cap \emptyset = \emptyset \tag{2.7a}$$
$$A \cup \Omega = \Omega; \qquad A \cap \Omega = A \tag{2.7b}$$
$$A \cup A^{\complement} = \Omega; \qquad A \cap A^{\complement} = \emptyset \tag{2.7c}$$
$$A \triangle B = (A \setminus B) \uplus (B \setminus A) \tag{2.7d}$$
$$A \setminus A = \emptyset \tag{2.7e}$$
$$A \triangle \emptyset = A; \qquad A \triangle A = \emptyset \tag{2.7f}$$
$$X \triangle A = X \setminus A \tag{2.7g}$$
$$A \cup B = (A \triangle B) \uplus (A \cap B) \tag{2.7h}$$
$$A \cap B = (A \cup B) \setminus (A \triangle B) \tag{2.7i}$$
$$A \triangle B = \emptyset \ \textit{ if and only if } \ B = A \tag{2.7j}$$

The proof is left as exercise 2.1.

Definition 2.8 (Power set)**.** The **power set**

$$2^{\Omega} := \mathfrak{P}(\Omega) := \{A : A \subseteq \Omega\}$$

of a set Ω is the set of all its subsets. $\square$

Remark 2.5. Note that $\emptyset \in 2^{\Omega}$ for any set Ω, even if $\Omega = \emptyset$: $2^{\emptyset} = \{\emptyset\}$. It follows that the power set of the empty set is not empty. $\square$

A lot more will be said about sets once families are defined.

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[4] $\mathbb{R}$ is the set of all real numbers, i.e., the kind of numbers that make up the x-axis and y-axis in a beginner's calculus course (see remark 2.6 ("Classification of numbers") on p.15).

[5] Neighborhoods of a point will be discussed in the chapter on the topology of $\mathbb{R}^n$ (see (10.6) on p.188) In short, the δ–neighborhood of a is the set of all points with distance less than δ from a.

2.2 Numbers

Remark 2.6 (Classification of numbers)**.** [6]

We call numbers without decimal points such as $3, -29, 0, 3000000, 3 \cdot 10^6, -1, \dots$ **integers** and we write $\mathbb{Z}$ for the set of all integers.

Numbers in the set $\mathbb{N} = \{1, 2, 3, \dots\}$ of all strictly positive integers are called **natural numbers**.

A number that is an integer or can be written as a fraction is called a **rational number** and we write $\mathbb{Q}$ for the set of all rational numbers. Examples of rational numbers are

$$\tfrac{3}{4},\ -0.75,\ -\tfrac{1}{3},\ .\bar{3},\ \tfrac{13}{4},\ -5,\ 2.99\bar{9},\ -37\tfrac{2}{7}.$$

The bar on top of the rightmost part of a decimal such as "$.\bar{3}$" means that this part should be repeated over and over again, e.g., $.\bar{3} = 0.33333333333\dots$ and $1.234\overline{567} = 1.234567567567\dots$.

Note that a mathematician does not care whether a rational number is written as a fraction "$\frac{numerator}{denominator}$" or as a decimal. The following all are representations of one third:

$$0.\bar{3} \ = \ .\bar{3} \ = \ 0.33333333333\dots \ = \ \frac{1}{3} \ = \ \frac{-1}{-3} \ = \ \frac{2}{6}, \tag{2.8}$$

and here are several equivalent ways of expressing the number minus four:

$$-4 \ = \ -4.000 \ = \ -3.\bar{9} \ = \ -\frac{12}{3} \ = \ \frac{4}{-1} \ = \ \frac{-4}{1} \ = \ \frac{12}{-3} \ = \ -\frac{400}{100}. \tag{2.9}$$

We call the barred portion of the decimal digits the **period** of the number and we also talk about **repeating decimals**. The number of digits in the barred portion is called the **period length**. This period length can be bigger than one. For example, the number $1.234\overline{567}$ from above has period length 3 and the number $0.1\overline{45}$ has period length 2.

If $q \in \mathbb{Q}$ then there are unique integers n and d such that $q = \frac{n}{d}$ and

- **a.** $d \in \mathbb{N}$,
- **b.** d is minimal: there are no numbers $n' \in \mathbb{Z}$ and $d' \in \mathbb{N}$ such that $q = \frac{n'}{d'}$ and $d' < d$.

We say that this choice of n and d is a representation of q in **lowest terms** or that q is then written in lowest terms. For example, the representation of $.\bar{3}$ in lowest terms is $\frac{1}{3}$ and the representation of -4 in lowest terms is $\frac{-4}{1}$.

Note that if $q \in \mathbb{Q}$ is strictly positive and if $\frac{d}{n}$ represents q in lowest terms then $d \in \mathbb{N}$.

There are numbers which cannot be expressed as integers or fractions or numbers with a finite amount of decimals to the right of the decimal point. Examples are $\sqrt{2}$ *and* $\boldsymbol{\pi}$. Those **irrational numbers** (that is their official name) fill the gaps between the rational numbers. In fact, there is a simple way (but not easy to prove) of characterizing irrational numbers: Rational numbers are those

[6] The classification of numbers in this section is not meant to be mathematically exact. For this consult, e.g., [1] B/G (Beck/Geoghegan).

that can be expressed with at most finitely many digits to the right of the decimal point, including repeating decimals. You can find the underlying theory and exact proofs in B/G ch.12. Irrational numbers must then be those with infinitely many decimal digits without any continually repeating patterns.

Example 2.3. To illustrate that repeating decimals are in fact rational numbers we convert $x = 0.1\overline{45}$ into a fraction:

$$99x \ = 100x - x \ = \ 14.5\overline{45} - 0.1\overline{45} = \ 14.4$$

It follows that $x = 144/990$, and that is certainly a fraction. $\square$

Now we can finally give an informal definition of the most important kind of numbers: We call any kind of number, either rational or irrational, a **real number** and we write $\mathbb{R}$ for the set of all real numbers. It can be shown that there are a lot more irrational numbers than rational numbers, even though $\mathbb{Q}$ is a **dense subset** in $\mathbb{R}$ in the following sense: No matter how small an interval $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ of real numbers you choose, it will contain infinitely many rational numbers. $\square$

Definition 2.9 (Types of numbers)**.** We summarize what was said sofar about the classification of numbers:

$\mathbb{N} := \{1, 2, 3, \dots\}$ denotes the set of **natural numbers**.

$\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, \dots\}$ denotes the set of all **integers**.

$\mathbb{Q} := \{n/d : n \in \mathbb{Z}, d \in \mathbb{N}\}$ denotes the set of all **rational numbers**.

$\mathbb{R} :=$ {all integers or decimal numbers with finitely or infinitely many decimal digits} denotes the set of all **real numbers**.

$\mathbb{R} \setminus \mathbb{Q} =$ {all real numbers which cannot be written as fractions of integers} denotes the set of all **irrational numbers**. There is no special symbol for irrational numbers. Example: $\sqrt{2}$ and π are irrational.

The following are customary abbreviations of some often referenced sets of numbers:

$\mathbb{N}_0 \ := \ \mathbb{Z}_+ \ := \mathbb{Z}_{\geq 0} \ := \{0, 1, 2, 3, \dots\}$ denotes the set of non–negative integers,

$\mathbb{R}_+ \ := \ \mathbb{R}_{\geq 0} \ := \ \{x \in \mathbb{R} : x \geqq 0\}$ denotes the set of all non–negative real numbers,

$\mathbb{R}^+ \ := \ \mathbb{R}_{>0} \ := \ \{x \in \mathbb{R} : x > 0\}$ denotes the set of all positive real numbers,

$\mathbb{R}^\star \ := \ \mathbb{R}_{\neq 0} \ := \ \{x \in \mathbb{R} : x \neq 0\}$. $\square$

Definition 2.10 (Intervals of real numbers)**.** We use the following notation for intervals of real numbers a, b:

$[a, b] := \{x \in \mathbb{R} : a \leqq x \leqq b\}$ is called the **closed interval** with endpoints a and b.

$]a, b[:= \{x \in \mathbb{R} : a < x < b\}$ is called the **open interval** with endpoints a and b.

$[a, b[:= \{x \in \mathbb{R} : a \leqq x < b\}$ and $]a, b] := \{x \in \mathbb{R} : a < x \leqq b\}$ are called **half-open intervals** with endpoints a and b.

We further define the following intervals of "infinite length":

$$(2.10)\qquad \begin{aligned}]-\infty,a] :=&\{x\in\mathbb{R}: x \leqq a\},\ \ ]-\infty,a[:= \{x\in\mathbb{R}: x<a\},\\ ]a,\infty[:=&\{x\in\mathbb{R}: x>a\},\ \ [a,\infty[:= \{x\in\mathbb{R}: x\geqq a\},\ \ [-\infty,\infty[:= \mathbb{R}\end{aligned}$$

Finally we define $[a,b[:=]a,b[:=]a,b] := \emptyset$ for $a \geqq b$ and $[a,b] := \emptyset$ for $a > b$. □

Definition 2.11 (Absolute value)**.** For a real number x we define its **absolute value** as

$$|x| = \begin{cases} x & \text{if } x \geqq 0, \\ -x & \text{if } x < 0. \end{cases} \square$$

□

Example 2.4. $|3| = 3; |-3| = 3; |-5.38| = 5.38$. □

Remark 2.7. For any real number x we have

$$(2.11)\qquad \sqrt{x^2} = |x|. \ \square$$

Assumption 2.1 (Square roots are always assumed non–negative)**.** Remember that for any number a it is true that

$$a\cdot a = (-a)(-a) = a^2 \qquad \text{e.g.,}\quad 2^2 = (-2)^2 = 4$$

or that, expressed in form of square roots, for any number $b \geqq 0$

$$(+\sqrt{b})(+\sqrt{b}) = (-\sqrt{b})(-\sqrt{b}) = b.$$

We will always assume that "$\sqrt{b}$" is the **positive** value unless the opposite is explicitly stated. Example: $\sqrt{9} = +3$, not -3. □

Proposition 2.2 (The Triangle Inequality for real numbers)**.** *The following inequality is used all the time in mathematical analysis to show that the size of a certain expression is limited from above:*

$$(2.12)\qquad Triangle\ Inequality : |\boldsymbol{a}+\boldsymbol{b}| \leqq |\boldsymbol{a}| + |\boldsymbol{b}|$$

This inequality is true for any two real numbers a and b.

It is easy to prove this: just look separately at the three cases where both numbers are non-negative, both are negative or where one of each is positive and negative. ■

Proposition 2.3 (The Triangle Inequality for n real numbers)**.** *The above inequality also holds true for more than two real numbers: Let $n \in \mathbb{N}$ such that $n \geqq 2$. Let $a_1, a_2, \dots, a_n \in \mathbb{N}$. Then*

$$(2.13)\qquad |a_1 + a_2 + \ldots + a_n| \leqq |a_1| + |a_2| + \ldots + |a_n|$$

The proof will be done by induction, a principle which is defined first:

Definition 2.12 (Principle of proof by mathematical induction)**.** Actually, "definition" is a misnomer. This principle is a mathematical statement that follows from the structure of the natural numbers which have a starting point to the "left" (a smallest element 1) and then progress in the well understood sequence [7]

$$2,\ 3,\ 4,\ \dots,\ k-1,\ k,\ k+1,\ \dots$$

This is the principle: Let us assume that we know that some statement can be proved to be true in the following two situations:

A. Base case. The statement is true for some (small) k_0; usually that means $k_0 = 0$ or $k_0 = 1$

B. Induction Step. We prove the following for all $k \in \mathbb{N}_0$ such that $k \geqq k_0$: if the property is true for k ("**Induction Assumption**") then it will also be true for $k+1$

C. Conclusion: Then the property is true for any $k \in \mathbb{N}_0$ such that $k \geqq k_0$. □

Either you have been explained this principle before and say "Oh, that – what's the big deal?" or you will be mighty confused. So let me explain how it works by walking you through the proof of the triangle inequality for n real numbers (2.13).

Proof of the triangle inequality for n real numbers:

A. For $k_0 = 2$, inequality 2.13 was already shown (see 2.12), so we found a k_0 for which the property is true.

B. Let us assume that 2.13 is true for some $k \geqq 2$. We now must prove the inequality for $k+1$ numbers $a_1, a_2, \dots, a_k, a_{k+1} \in \mathbb{N}$: We abbreviate

$$A \ :=\ a_1 + a_2 + \ldots + a_k; \qquad B \ :=\ |a_1| + |a_2| + \ldots + |a_k|$$

then our induction assumption for k numbers is that $|A| \leqq B$. We know the triangle inequality is valid for the two variables A and a_{k+1} and it follows that $|A + a_{k+1}| \leqq |A| + |a_{k+1}|$. Look at both of those inequalities together and you have

$$|A + a_{k+1}| \leqq |A| + |a_{k+1}| \leqq B + |a_{k+1}| \tag{2.14}$$

In other words,

$$\big|\big(a_1 + a_2 + \ldots + a_k\big) + a_{k+1}\big| \ \leqq\ B + |a_{k+1}| \ =\ \big(|a_1| + |a_2| + \ldots + |a_k|\big) + |a_{k+1}| \tag{2.15}$$

and this is (2.13) for $k+1$ rather than k numbers: We have shown the validity of the triangle inequality for $k+1$ items under the assumption that it is valid for k items. It follows from the induction principle that the inequality is valid for any $k \geqq k_0 = 2$. ■

To summarize what we did in all of part B: We were able to show the validity of the triangle inequality for $k+1$ numbers under the assumption that it was valid for k numbers.

[7] The first two chapters of [1] B/G (Beck/Geoghegan) use the "axiomatic" method to develop the mathematical structure of integers and natural numbers and give an exact proof of the induction principle.

Remark 2.8 (Why induction works)**.** But how can we from all of the above conclude that the triangle inequality works for all $n \in \mathbb{N}$ such that $n \geqq k_0 = 2$? That's much simpler to demonstrate than what we just did.

Step 1:	We know that it is true for $k_0 = 2$ because that was actually proved in A.
Step 2:	But according to B, if it is true for k_0, it is also true for the successor $k_0 + 1 = 3$.
Step 3:	But according to B, if it is true for $k_0 + 1$, it is also true for the successor $(k_0 + 1) + 1 = 4$.
Step 4:	But according to B, if it is true for $k_0 + 2$, it is also true for the successor $(k_0 + 2) + 1 = 5$.
......	....
Step $53,920$:	But according to B, if it is true for $k_0 + 53,918$, it is also true for the successor $(k_0 + 53,918) + 1 = 53,921$.
......	....

And now you understand why it is true for any natural number $n \geqq k_0$. $\square$

2.2.1 Rings and Algebras of Sets ($\star$)

This section is optional. You will benefit from examining the proof of prop.2.4 on p.19 and learn how to split a proof which involves 3 or 4 sets can be split into easily dealt with cases.

Definition 2.13 (Rings and Algebras of Sets)**.** A subset $\mathscr{R}$ of 2^Ω (a set of sets!) is called a **ring of sets** if it is closed with respect to the operations "$\cup$" and "$\setminus$", i.e.,

$$R_1 \cup R_2 \in \mathscr{R} \text{ and } R_1 \setminus R_2 \in \mathscr{R} \quad \text{whenever } R_1, R_2 \in \mathscr{R}. \tag{2.16}$$

A subset $\mathscr{A}$ of 2^Ω is called an **algebra of sets** if $\Omega \in \mathscr{A}$ and $\mathscr{A}$ is a ring of sets. $\square$

Proposition 2.4. ***1.*** *Let $\mathscr{R}$ be a ring of sets and $A, B \in \mathscr{R}$. Then $A \triangle B \in \mathscr{R}$ and $A \cap B \in \mathscr{R}$.*

2. *Let $\mathscr{R}$ be a ring of sets and $A, B, C \in \mathscr{R}$. Then*

a.	$(A \triangle B) \triangle C = A \triangle (B \triangle C)$	*(associativity of $\triangle$)*
b.	$A \triangle \emptyset = \emptyset \triangle A = A$	*(neutral element $\emptyset$ for $\triangle$)*
c.	$A \triangle A = \emptyset$	*(inverse element $\emptyset$ for $\triangle$)* [8]
d.	$A \triangle B = B \triangle A$	*(commutativity of $\triangle$)*

Further we have the following for the intersection operation:

e.	$(A \cap B) \cap C = A \cap (B \cap C)$	*(associativity of $\cap$)*
f.	$A \cap \Omega = \Omega \cap A = A$	*(neutral element Ω for $\cap$)*
g.	$A \cap B = B \cap A$	*(commutativity of $\cap$)*

And we have the following interrelationship between $\triangle$ and $\cap$:

[8] The inverse element for A in the sense of def.**??** on p.**??**. is A itself!

h. $A \cap (B \triangle C) \;=\; (A \cap B) \triangle (A \cap C)$ *(distributivity)*

Solution:

*Proof of **a.** The proof given here is based on with set membership. It is very tedious and there is a much more elegant proof but it requires knowledge of indicator functions* [9] *and of base* 2 *modular arithmetic (see, e.g., [1] B/G (Beck/Geoghegan) ch.6.2).*

By definition $x \in U \triangle V$ if and only if either $x \in U$ or $x \in V$, i.e.,
(either) $\left[\ x \in U \text{ and } x \notin V\ \right]$ or $\left[\ x \in V \text{ and } x \notin U\ \right]$

Hence $x \in (A \triangle B) \triangle C$ means either $x \in (A \triangle B)$ or $x \in C$, i.e.,
either $\left[\ x \in A, x \notin B \text{ or } x \in B, x \notin A\ \right]$ or $x \in C$, i.e., we have one of the following four combinations:

a.	$x \in A$	$x \notin B$	$x \notin C$
b.	$x \notin A$	$x \in B$	$x \notin C$
c.	$x \in A$	$x \in B$	$x \in C$
d.	$x \notin A$	$x \notin B$	$x \in C$

and $x \in A \triangle (B \triangle C)$ means either $x \in A$ or $x \in (B \triangle C)$, i.e.,
either $x \in A$ or $\left[\ x \in B, x \notin C \text{ or } x \in C, x \notin B\ \right]$, i.e., we have one of the following four combinations:

1.	$x \in A$	$x \in B$	$x \in C$
2.	$x \in A$	$x \notin B$	$x \notin C$
3.	$x \notin A$	$x \in B$	$x \notin C$
4.	$x \notin A$	$x \notin B$	$x \in C$

*We have a perfect match **a** ↔ **2**, **b** ↔ **3**, **c** ↔ **1**, **d** ↔ **4**. and this completes the proof of **a.***

■

Remark 2.9 (Rings of sets as rings)**.** A set $G = (G, \oplus)$ with a "binary operation" $g_3 = g_1 \oplus g_2$ – i.e., an operation $\oplus$ which assigns to any two $g_1, g_2 \in G$ a third element $g_3 \in G$ – such that $\oplus$ satisfies **a** – **d** above with $\triangle$ playing the role of $\oplus$ is called an abelian or commutative group (see ch.**??** on p.**??**.)

A set $R = (R, \oplus, \odot)$ with two binary operations $g_3 = g_1 \oplus g_2$ and $g_4 = g_1 \odot g_2$ which satisfies **a** – **h** above with $\triangle$ playing the role of $\oplus$ and $\cap$ playing the role of $\odot$ is called a commutative ring with multiplicative unit. $\oplus$ is customarily referred to as "addition" and $\odot$ is customarily referred to as "multiplication". This explains the name "ring of sets" for $\mathscr{R} \;=\; (\mathscr{R}, \triangle, \cap)$.

We note that rings of sets satisfy axioms 1.1 – 1.4 but not axiom 1.5 (the cancellation axiom) of [1] B/G (Beck/Geoghegan) ch.1.1. This explains the name "ring of sets".

The name "algebra of sets" for a ring of sets which contains Ω stems from the fact that such systems of subsets of Ω are "boolean algebras". □

[9] Indicator functions will be discussed in ch.6.2 on p.111 and in ch.8.4 on p.145.

2.3 Exercises for Ch.2

2.3.1 Exercises for Sets

Exercise 2.1. Prove the set identities of prop.2.1.

Exercise 2.2. Let $X = \{x, y, \{x\}, \{x, y\}\ \}$. True or false?

a. $\{x\} \in X$ **c.** $\{\ \{x\}\ \} \in X$ **e.** $y \in X$ **g.** $\{y\} \in X$
b. $\{x\} \subseteq X$ **d.** $\{\ \{x\}\ \} \subseteq X$ **f.** $y \subseteq X$ **h.** $\{y\} \subseteq X$ $\square$

For the subsequent exercises refer to example 4.4 for the preliminary definition of cardinality of a set and to def.4.1 (Cartesian Product of two sets) for the definition of Cartesian product. You find both in ch.4.1 (Cartesian products and relations) on p.69

Exercise 2.3. Find the cardinality of each of the following sets:

a. $A = \{x, y,\ \{x\},\ \{x, y\}\ \}$ **c.** $C = \{u, v, v, v, u\}$ **e.** $E = \{\sin(k\pi/2) : k \in \mathbb{Z}\}$
b. $B = \{1,\ \{0\},\ \{1\}\ \}$ **d.** $D = \{3z - 10 : z \in \mathbb{Z}\}$ **f.** $F = \{\pi x : x \in \mathbb{R}\}$ $\square$

Exercise 2.4. Let $X = \{x, y, \{x\}, \{x, y\}\ \}$ and $Y = \{x, \{y\}\ \}$. True or false?

a. $x \in X \cap Y$ **c.** $x \in X \cup Y$ **e.** $x \in X \setminus Y$ **g.** $x \in X \Delta Y$
b. $\{y\} \in X \cap Y$ **d.** $\{y\} \in X \cup Y$ **f.** $\{y\} \in X \setminus Y$ **h.** $\{y\} \in X \Delta Y$ $\square$

Exercise 2.5. Let $X = \{1, 2, 3, 4\}$ and let $Y = \{x, y\}$.

a. What is $X \times Y$? **c.** What is $\text{card}(X \times Y)$? **e.** Is $(x, 3) \in X \times Y$? **g.** Is $3 \cdot x \in X \times Y$?
b. What is $Y \times X$? **d.** What is $\text{card}(Y \times X)$? **f.** Is $(x, 3) \in Y \times X$? **h.** Is $2 \cdot y \in Y \times X$? $\square$

3 Logic (Skim this!)

This chapter uses material presented in ch.2 (Logic) and ch.3 (Methods of Proofs) of [4] Bryant, Kirby Course Notes for MAD 2104.

3.1 Prologue: Notation for Functions

The material on functions presented in this section will be discussed again and in greater detail in chapter 4 (unctions and relations) on p.69. It is presented at the beginning of this chapter about logic because the definition of a function is needed to properly discuss statement functions (see ch.3.2 on p.25).

Note 3.1 (Motivation for a good function definition)**.** When discussing logic we deal with statement functions (predicates) (see def.3.4 on p.25) and we are in the same predicament as when discussing some run of the mill functions known from calculus such as $f_1(x) = \sqrt{x}$ and $f_2(x,y) = \ln(x-y)$: Sometimes $f_1(x)$ means the entire graph, i.e., the entire collection of pairs $\big(x, \sqrt{x}\big)$ and sometimes it just refers to the function value $\sqrt{x}$ for a "fixed but arbitrary" number x. In case of the function $f_2(x)$: Sometimes $f_2(x,y)$ means the entire graph, i.e., the entire collection of pairs $\big((x,y), \ln(x-y)\big)$ and sometimes it just refers to the function value $\ln(x-y)$ for a pair of "fixed but arbitrary" numbers (x,y).

This issue is addressed in the material of ch.4.2 on p.72 which precedes the mathematically precise definition of a function (def.4.6 on p.75). You are encouraged to look at it once you have read the remainder of this short section as ch.4.2 contains everything you see here.

To get to a usable definition of a function there are several things to consider. In the following $f_1(x)$ and $f_2(x,y)$ again denote the functions $f_1(x) = \sqrt{x}$ and $f_2(x,y) = \ln(x-y)$.

a. The source of all allowable arguments (x–values in case of $f_1(x)$ and (x,y)–values in case of $f_2(x,y)$) will be called the domain of the function. The domain is explicitly specified as part of a function definition and it may be chosen for whatever reason to be only a subset of all arguments for which the function value is a valid expression. In case of the function $f_1(x)$ this means that the domain must be restricted to a subset of the interval $[0,\infty[$ because the square root of a negative number cannot be taken. In case of the function $f_2(x,y)$ this means that the domain must be restricted to a subset of $\{\ (x,y) : x,y \in \mathbb{R}\ \mathit{and}\ x-y>0\}$ because logarithms are only defined for strictly positive numbers.

b. The set to which all possible function values belong will be called the codomain of the function. As is the case for the domain, the codomain also is explicitly specified as part of a function definition. It may be chosen as any <u>superset</u> of the set of all function values for which the argument belongs to the domain of the function. In case of the function $f_1(x)$ this means that we are OK if the codomain is a superset of the interval $[0,\infty[$. Such a set is big enough because square roots are never negative. It is OK to specify the interval $]-3.5,\infty$ or even the set $\mathbb{R}$ of all real numbers as the codomain. In case of the function $f_2(x,y)$ this means that we are OK if the codomain contains $\mathbb{R}$. Not that it would make a lot of sense but the set $\mathbb{R} \cup \{$ all inhabitants of Chicago $\}$ also is an acceptable choice for the codomain.

c. A function $y = f(x)$ is not necessarily something that maps (assigns) numbers or pairs of numbers to numbers but domain and codomain can be a very different kind of animal. In this chapter on logic you will learn about statement functions $A(x)$ which assign arguments x from an arbitrary set $\mathscr{U}$, called the universe of discourse, to statements $A(x)$, i.e., sentences that are either true or false.

d. Considering all that was said so far you can now think of the graph of a function $f(x)$ with domain D and codomain C (see the beginning of this chapter) as the set $\Gamma_f := \{(x, f(x)) : x \in D\}$. Alternatively you can characterize this function by the assignment rule which specifies how the function value $f(x)$ depends on any given argument $x \in D$. We write "$x \mapsto f(x)$ to indicate this. You can also write instead $f(x) =$ whatever the actual function value will be. This is possible if you do not write about functions in general but about specific functions such as $f_1(x) = \sqrt{x}$ and $f_2(x, y) = \ln(x - y)$. We further write "$f : C \to D$" as a short way of saying that the function $f(x)$ has domain C and codomain D.

In case of the function $f_1(x) = \sqrt{x}$ for which we choose the interval $X := [2.5, 7]$ as the domain (small enough because $X \subseteq [0, \infty[$) and $Y :=]1, 3[$ as the codomain (big enough because $1 < \sqrt{x} < 3$ for any $x \in X$) we specify this function as

$$\text{either } f_1 : X \to Y, \quad x \mapsto \sqrt{x} \qquad\qquad \text{or } f_1 : X \to Y, \quad f(x) = \sqrt{x}.$$

Let us choose $U := \{(x, y) : x, y \in \mathbb{R} \;\; 1 \leqq x \leqq 10 \text{ and } y < -2\}$ as the domain and $V := [0, \infty[$ as the codomain for $f_2(x, y) = \ln(x - y)$. These choices are OK because $x - y \geqq 1$ for any $(x, y) \in U$ and hence $ln(x - y) \geqq 0$, i.e., $f_2(x, y) \in V$ for all $(x, y \in U$. We specify this function as

$$\text{either } f_2 : U \to V, \quad (x, y) \mapsto \ln(x - y) \qquad\qquad \text{or } f_2 : U \to V, \quad f(x, y) = \ln(x - y). \;\; \square$$

We incorporate the above into the following preliminary definition.

Definition 3.1 (Preliminary definition of a function)**.** A **function** f consists of two nonempty sets X and Y and an assignment rule $x \mapsto f(x)$ which assigns any $x \in X$ uniquely to some $y \in Y$. We write $f(x)$ for this assigned value and call it the function value of the argument x. X is called the **domain** and Y is called the **codomain** of f. We write

$$f : X \to Y, \qquad x \mapsto f(x). \tag{3.1}$$

We read "$a \mapsto b$" as "a is assigned to b" or "a maps to b" and refer to $\mapsto$ as the **maps to operator** or **assignment operator**. The **graph** of such a function is the collection of pairs

$$\Gamma_f := \{(x, f(x)) : x \in X\}. \tag{3.2}$$

$\square$

Remark 3.1. The name given to the argument variable is irrelevant. Let f_1, f_2, X, Y, U, V be as defined in **d** of note 3.1. The function

$$g_1 : X \to Y, \quad p \mapsto \sqrt{p}$$

is identical to the function f_1. The function

$$g_2 : U \to V, \quad (t, s) \mapsto \ln(t - s)$$

is identical to the function f_2 and so is the function

$$g_3 : U \to V, \quad (s, t) \mapsto \ln(s - t).$$

The last example tells you that you can swap function names as long as you do it consistently in all places. □

Note 3.2 (Textual variables)**.** It was mentioned in **c** above that the input variables and function values need not necessarily numbers but they can also be textual. For example, the domain of a function may consist of the first names of certain persons.

A note on textual variables: If the variable is the last name of the person James Joice and valid input for the function $F : p \mapsto$ "Each morning p writes two pages.") then we write interchangeably Joyce or 'Joyce'. Quotes are generally avoided unless they add clarity.

In the above example "Each morning 'Joyce' writes two pages." emphasizes that Joyce is the replacement of a parameter whereas F('Joyce') does not seem to improve the simpler notation F(Joyce) and you will most likely see the expression F(Joyce) = "Each morning 'Joyce' writes two pages." □

We also need the definition of a cartesian product. [10]

Definition 3.2 (Preliminary definition: cartesian product)**.** Let X and Y be two sets The set

$$X \times Y \; := \{(x, y) : x \in X, y \in Y\} \tag{3.3}$$

is called the **cartesian product** of X and Y.

Note that the order is important: (x, y) and (y, x) are different unless $x = y$.

We write X^2 as an abbreviation for$X \times X$.

This definition generalizes to more than two sets as follows: Let $X_1, X_2, \dots, X_n$ be sets. The set

$$X_1 \times X_2 \cdots \times X_n \; := \{(x_1, x_2, \dots, x_n) : x_j \in X_j \text{ for each } j = 1, 2, \dots n\} \tag{3.4}$$

is called the cartesian product of $X_1, X_2, \dots, X_n$.

We write X^n as an abbreviation for$X \times X \times \cdots \times X$. □

Example 3.1. The graph Γ_f of a function with domain X and codomain Y (see def.3.2) is a subset of the cartesian product$X \times Y$. □

Example 3.2. The domains given in **a** and **d** of note 3.1 are subsets of the cartesian product $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : x, y \in \mathbb{R}\}$. □

[10] See ch.4.1 (Cartesian products and relations) on p.69 for the real thing and examples.

3.2 Statements and Statement Functions

Definition 3.3 (Statements)**.** A **statement** [11] is a sentence or collection of sentences that is either true or false. We write T or **true** for "true" and F or **false** for "false" and we refer to those constants as **truth values** □

Example 3.3. The following are examples of statements:

- **a.** "Dogs are mammals" (a true statement);
- **b.** "Roses are mammals. 7 is a number." This is a false statement which also could have been written as a single sentence: "Roses are mammals and 7 is a number";
- **c.** "I own 5 houses" (a statement because this sentence is either true or false depending on whether I told the truth or I lied);
- **d.** "The sum of any two even integers is even" (a true statement);
- **e.** "The sum of any two even integers is even **and** Roses are mammals" (a false statement);
- **f.** "**Either** the sum of any two even integers is even **or** Roses are mammals" (a true statement). □

Example 3.4. The following are **not** statements:

- **a.** "Who is invited for dinner?"
- **b.** "$2x = 27$" (the variable x must be bound (specified) to determine whether this sentence is true or false: It is true for $x = 13.5$ and it is false for $x = 33$)
- **c.** "$x^2 + y^2 = 34$" (both variables x and y must be bound to determine whether this sentence is true or false It is true for $x = 5$ and $y = 3$ and it is false for $x = 7.8$ and $y = 2$)
- **d.** "Stop bothering me!" □

For the remainder of the entire chapter on logic we define

$$\mathscr{S} := \text{the set of all statements} \tag{3.5}$$

$\mathscr{S}$ will appear as the codomain of statement functions.

Be sure to understand the material of ch.3.1 (Prologue: Notation for Functions) on p.22) before continuing.

Definition 3.4 (Statement functions (predicates))**.** We need to discuss some preliminaries before arriving at the definition of a statement function. Let A be a sentence or collection of sentences which contains one or more variables (placeholders) such that, if each of those variables is assigned a specific value, it is either true or false, i.e., it is an element of the set $\mathscr{S}$ of all statements. If A contains n variables $x_1, x_2, \dots, x_n$ and if they are **bound**, i.e., assigned to the specific values $x_1 = x_{10},\ x_2 = x_{20}, \dots, x_n = x_{n0}$, we write $A(x_{10}, x_{20}, \dots, x_{n0})$ for the resulting statement.

To illustrate this let A := "x is green and y and z like each other". If we know the specific values for the variables x, y, z then this sentence will be true or false. For example $A(\text{this lime, Tim, Fred})$ is true or false depending on whether Tim and Fred do or do not like each other.

There are restrictions for the choice of $x_1 = x_{10}, x_2 = x_{20}, \dots, x_n = x_{n0}$: Associated with each variable x_j in A is a set $\mathscr{U}_j$ which we call the **universe of discourse**, in short, **UoD**, for the jth

[11] usually called a **proposition** in a course on logic but we do not use this term as in mathematics "proposition" means a theorem of lesser importance.

variable in A. Each value x_{j0} $(j = 1, 2, \dots n)$ must be chosen in such a way that $x_{j0} \in \mathscr{U}_j$. If this is not the case then the expression $A(x_{10}, x_{20}, \dots, x_{n0})$ is called **inadmissible** and we refuse to deal with it.

What was said can be rephrased as follows: We have an assignment $(x_1, x_2, \dots, x_n) \mapsto A(x_1, x_2, \dots, x_n)$ which results in a statement, i.e., an element of $\mathscr{S}$ (see (3.5)) just as long as $x_{j0} \in \mathscr{U}_j$. In other words we have a function

$$A : \mathscr{U}_1 \times \mathscr{U}_2 \times \cdots \times \mathscr{U}_n \to \mathscr{S}, \quad (x_1, x_2, \dots, x_n) \mapsto A(x_1, x_2, \dots, x_n) \tag{3.6}$$

in the sense of def. 3.1. with the cartesian product of the UoDs for $x_1, \dots, x_n$ as domain and $\mathscr{S}$ as codomain. We call such a function a **statement function** [12] or **predicate**. □

Note 3.3 (Relaxed notation for statement functions)**.** You should remember that a statement function is a function in the sense of def.3.1 but we we will often use the simpler notation

> A := "some text that contains the placeholders $x_1, x_2, \dots, x_n$ and evaluates to **true** or **false** once all x_j are bound"

together with the specification of each UoD $\mathscr{U}_j$ rather than the formal notation

$$A : \mathscr{U}_1 \times \mathscr{U}_2 \times \cdots \times \mathscr{U}_n \to \mathscr{S}, \quad (x_1, x_2, \dots, x_n) \mapsto A(x_1, x_2, \dots, x_n).$$

If A contains two or more variables then the formal notation has an advantage. There is no doubt when looking at an evaluation such as $A(5.5, 7, -3, 8)$ which placeholder in the string corresponds to 5.5, which one corresponds to 7 etc. When employing the relaxed notation then we decide this according to the following

> **Left to right rule for statement functions**: If the string A contains n different place holders then the expression $A(x_{10}, x_{20}, \dots x_{n0})$ implies the following: If the name of the first (leftmost) place holder in A is x then each occurrence of x is bound to the value x_{10}. If the name of the first of the remaining place holders in A is y then each occurrence of y is bound to the value x_{20}, ... After $n - 1$ steps the remaining placeholders all have the same name, say z and each occurrence of z is bound to the value x_{n0}. If there is any confusion about what is first, what is second, ... then this will be indicated when A is specified or when its variables are bound for the first time.

Example 3.5. In def.3.4 A = "x is green and y and z like each other" was used to illustrate the concept of a statement function. We never showed how to write the actual statement function. We must decide the UoDs for x, y, z and we define them as follows.
UoD for x: $\mathscr{U}_x$:= all plants and animals in the U.S.,
UoDs for y and z: $\mathscr{U}_y := \mathscr{U}_z :=$ all BU majors in actuarial science.

a. Here is the formal definition: Let A be the statement function

$$A : \mathscr{U}_x \times \mathscr{U}_y \times \mathscr{U}_z \to \mathscr{S}, \quad (x, y, z) \mapsto A(x, y, z) := \text{"}x \text{ is green and } y \text{ and } z \text{ like each other"}$$

[12] A statement function is usually called a **proposition function** in a course on logic. As previously mentioned, we do not use the term "proposition" in this document because in most brances of mathematics it refers to a theorem of lesser importance.

b. Here is the relaxed definition: Let A be the statement function

A := "x is green and y and z like each other" with UoDs $\mathscr{U}_x$ for x, $\mathscr{U}_y$ for y and $\mathscr{U}_z$ for z. $\square$

The example above and all those below for statement functions of more than a single variable employ the left to right rule. $\square$

Adhering to the left to right rule is not a big deal because of the following convention:

> *We will restrict ourselves in this document from now on to statement functions of one or two variables.*

Example 3.6. Let $A(t)$ = "$t - 4.7$ is an integer". Then $A : \mathbb{R} \to \mathscr{S},\ x \mapsto A(x)$ is a one parameter statement function with UoD $\mathbb{R}$ and x as the variable. Note that it is immaterial that we wrote t in the equation and x in the "$\mapsto$" expression because with deal with a dummy variable and we have employed its name consistently in both cases. We have

- **a.** $A(\text{Honda})$ = "'Honda' $- 4.7$ is an integer" is inadmissible because a car brand is not part of our universe of discourse.
- **b.** If $u_0 \in \mathscr{U}$ then $A(u_0)$ = "$u_0 - 4.7$ is an integer" is a <u>statement</u> which evaluates to true or false depending on that fixed but unknown value of u_0.
- **c.** If $n \in \mathscr{U}$ then $A(n)$ is the statement(!) "$n - 4.7$ is an integer". It does not matter that this expression looks exactly like the original A: The expression $A(n)$ implies that the parameter inside the sentence collection A which happens to be named "n" has been bound to a fixed (but unspecified) value also denoted by n. $\square$

Example 3.7. Let $B(x, y)$:= "$x^2 - y + 2 = 11$". Then $B : \mathbb{R}\times]1, 100[\ \to \mathscr{S},\ (x, y) \mapsto B(x, y)$ is a two parameter statement function with UoD $\mathbb{R}$ for x and UoD $]1, 100[$ for the variable y. Then

- **a.** $B(4, -2)$ = "$4^2 - (-2) + 2 = 11$" (a false statement) because x is the leftmost item in B.
- **b.** $B(z, 10)$ = "$z^2 - 10 + 2 = 11$" (true or false depending on z).
- **c.** **BE CAREFUL:** If $x, y \in \mathbb{R}$ then $B(y, x)$ = "$y^2 - x + 2 = 11$" and **NOT** "$x^2 - y + 2 = 11$" because the "evaluate left to right" rule matters, not any similarity or even coincidence between the symbols inside the sentence collection and in the evaluation $B(\cdot, \cdot)$ $\square$

Example 3.8. The following are predicates:

- **a.** P := "$2x = 27$" (see example 3.4.**b**), UoD $\mathscr{U} := \{x \in \mathbb{R} : x > 10\}$
- **b.** Q := "$x^2 + y^2 = 34$" (example 3.4.**c**), UoD $\mathscr{V} := \{(x, y) : x, y \in \mathbb{R} \text{ and } x < y\}$
- **c.** R := "$x^2 + y^2 = 34$ and $xy > 100$", UoDs are $\mathscr{W}_x := \mathscr{W}_y := [-50, 25]$.

Note the following for **c**: $R(-30, 20)$ evaluates to a false statement because $(-30) \cdot 20 > 100$ is false. $R(30, 20)$ does not evaluate to any kind of statement: It is an inadmissible expression because $30 \notin \mathscr{W}_x$.

d. The sentence "Stop bothering x!" is **not** a statement function because this imperative will not be true or false even if x is bound to a specific value. $\square$

Example 3.9. Let B := "$x + 7 = 16$ and d is a dog". Let $\mathscr{U}_x := \mathbb{N}$ and $\mathscr{U}_d := \{d : d \text{ is a vegetable or animal}\}$.

B becomes a statement function of two variables x and d if we specify that the UoD for x is $\mathscr{U}_x$ and the UoD for d is $\mathscr{U}_d$

Assume for the following that Robby is an animal.

a. $B(9, \text{Robby})$ is the statement "$9 + 7 = 16$ and Robby is a dog". It is true in case Robby is a dog and false in case Robby is not a dog.
b. $B(20, \text{Robby})$ is the statement "$20 + 7 = 16$ and Robby is a dog" which is false regardless of what Robby might be because $20 + 7 = 16$ by itself is false.
c. $B(d, F)$ is the statement "$d + 7 = 16$ **and** $F$is a dog": which is true or false depending on the fixed but unspecified values of d and F. Note that d corresponds to the leftmost variable x inside B and not to the second variable d!
d. $B(x)$ is not a valid expression as we do not allow "partial evaluation" of a predicate. [13]
□

3.3 Logic Operations and their Truth Tables

We now resume our discussion of statements.

3.3.1 Overview of Logical Operators

*Statements can be connected with **logical operators**, also called **connectives**, to form another statement, i.e., something that is either **true** or **false**.*

Here is an overview of the important connectives. [14] *Their meaning will be explained subsequently, once we define compound statements and compound statement functions.*

negation:	$\neg A$	***not*** A
conjunction:	$A \wedge B$	A ***and*** B
double arrow (biconditional):	$A \leftrightarrow B$	A ***double arrow*** B
logical equivalence:	$A \Leftrightarrow B$	A ***if and only if*** B
disjunction (inclusive or):	$A \vee B$	A ***or*** B
exclusive or:	A ***xor*** B	***either*** A ***or*** B, ***exactly one of*** A ***or*** B
arrow:	$A \to B$	A ***arrow*** B, ***if*** A ***then*** B
implication:	$A \Rightarrow B$	A ***implies*** B, ***if*** A ***then*** B

[13] To indicate that we consider d as fixed but arbitrary and want to interpret "$x + 7 = 16$ and d is a dog" as a statement function of only x as a variable we could have introduced the notation $B(\cdot, d) : x \mapsto B(x, d)$. Similarly, to indicate that we consider x as fixed but arbitrary and want to interpret "$x + 7 = 16$ and d is a dog" as a statement function of only d as a variable we could have introduced the notation $B(x, \cdot) : d \mapsto B(x, d)$. We choose not to overburden the reader with this additional notation. Rather, this situation can be handled by defining two new predicates $C : x \mapsto C(x) :=$ "$x + 7 = 16$ and z is a dog" and $D : d \mapsto D(d) :=$ "$z + 7 = 16$ and d is a dog" and then state that z is not a variable but a fixed (but unspecified) value.

[14] This order is rather unusual in that usually you would discuss biconditional and logical equivalence operators last, but logical equivalence between two statements A and B is what we think of when saying "A if and only if B" and it helps to understand what this phrase means in the context of logic as early as possible.

Notations 3.1 (use of symbols vs descriptive English)**.**

a. In the entire chapter on logic we generally use for logical operators their symbols like "$\neg$" or "$\Rightarrow$" in formulas but we use their corresponding English expressions (**not** and **implies** in this case) in connection with constructs which contain English language.

For example we would write $\neg(A \vee \neg B)$ rather than **not**$(A$ **or not** $B)$ but we would write "$d + 7 = 16$ **and** F is a dog" rather than "$(d + 7 = 16) \wedge (F$is a dog)"

b. Outside chapter 3 symbols are not used at all for logical operators. We use boldface such as "**and**" rather than just plain type face only to make it visually easier to understand the structure of a mathematical construct which employs connectives. □

Definition 3.5 (Compound statements)**.** A statement which does not contain any logical operators is called a **simple statement** and one that employs logical operators is called a **compound statement**.

Similarly statement functions which contain logical operators are called **compound statement functions**. □

Example 3.10. Statements **e** and **f** of example 3.3 are examples of compound statements.

In **e** the two simple statements "The sum of any two even integers is even" and "Roses are mammals" are connected by **and**.

In **f** the two simple statements "The sum of any two even integers is even" and "Roses are mammals" are connected by **either** ... **or**. □

3.3.2 Negation and Conjunction, Truth Tables and Tautologies (Understand this!)

We now give the definition of the first two logical operators which were introduced in the table of section 3.3.1.

Definition 3.6 (Negation)**.** The **negation operator** is represented by the symbol "$\neg$" and it reverses the truth value of a statement A, i.e., if A is **true** then $\neg(A)$ is **false** and if A is **false** then $\neg(A)$ is **true**.

(3.7) This is expressed in this "truth table" for $\neg A$:[15]

A	¬ A
F	T
T	F

□

Example 3.11. Let A := "Rover is a horse". Then $\neg A$ = "Rover is **not** a horse" and $\neg\neg A = \neg(\neg A)$ = "Rover is a horse" = A.

Let us not quibble here about whether $\neg\neg A$ is not in reality the statement "Rover is not not a horse" which admittedly means the same as "Rover is a horse" but looks different.

There is no question about the fact that the T/F values for A and $\neg\neg A$ are the same. Just compare column 1 with column 3.

A	$\neg A$	$\neg(\neg A)$
F	T	F
T	F	T

[15]The definition of a truth table will be given shortly. See def.3.8 on p.30.

Note that we did not use any specifics about A. We derived the T/F values for $\neg\neg A$ from those in the second column by applying the definition of the $\neg$ operator to the statement $B := \neg A$.

In other words we have proved that the statements A and $\neg\neg A$ are **logically equivalent** in the sense that one of them is true whenever the other one is true and vice versa. □

*All operators discussed subsequently are **binary operators**, i.e., they connect two input parameters (statements) A, B and four rather than two rows are needed to show what will happen for each of the four combinations A: **false** and B: **false**, A: **false** and B: **true**, A: **true** and B: **false**, A: **true** and B: **true**.*

*In contrast, the already discussed negation operator "¬" is a **unary operators**, i.e., it has a single input parameter. We will keep referring to "¬" as a connective even though there are no two or more items that can be connected.*

Definition 3.7 (Conjunction)**.** The **conjunction operator** is represented by the symbols "$\wedge$" or "**and**". The expression A **and** B is **true** if and only if both A and B are **true**.

(3.8) Truth table for A **and** B:

A	B	$A \wedge B$
F	F	F
F	T	F
T	F	F
T	T	T

The **and** connective generalizes to more than two statements $A_1, A_2, \ldots, A_n$ in the obvious manner:

$A_1 \wedge A_2 \wedge \cdots \wedge A_n$ is **true** if and only if each one of $A_1, A_2, \ldots, A_n$ is **true** and **false** otherwise. □

Definition 3.8 (Truth table)**.** A **truth table** contains the symbols for statements in the header, i.e., the top row and shows in subsequent rows how their truth values relate.

It contains in the leftmost columns statements which you may think of as varying inputs and it contains in the columns to the right compound statements which were built from those inputs by the use of logical operators. We have a row for each possible combination of truth values for the input statements. Such a combination then determines the truth value for each of the other statements.

When we count rows we start with zero for the header which contains the statement names. Row 1 is the first row which contains T/F values.

An example for a truth table is the following table which you encountered in the definition above 3.7 of the conjunction operator:

A	B	$A \wedge B$
F	F	F
F	T	F
T	F	F
T	T	T

Here the input statements are A and B. The compound statement $A \wedge B$ is built from those inputs with the use of the $\wedge$ operator. We have 4 possible T/F combinations for A and B and each one of those determines the truth value of $A \wedge B$. For example, row 2 contains A:F and B:T and from this we obtain F as the corresponding truth value of $A \wedge B$.

Some truth tables have more than two inputs. If there are three statements A, B, C from wich the compound statements that interest us are built then there will be $2^3 = 8$ rows to hold all possible combinations of truth values and for n inputs there will be 2^n rows. □

Definition 3.9 (Logically impossible)**.** The statements A and B in the truth table of def.3.8 were of a generally nature and all four T/F combinations had to be considered. If we deal with statements which are more specific but have some variability because they contain place holders [16] then there may be dependencies that rule out certain combinations as nonsensical. For example let x be some fixed but unspecified number and look at a truth table which has the statements $A := A(x) :=$ "$x > 5$" and $B := B(x) :=$ "$x > 7$" as input. It is clearly impossible that A is false and B is true, no matter what value x may have.

We call such combinations **logically impossible** or **contradictory**. We abbreviate "logically impossible" with **L/I**.

Both truth tables indicate that the combination A:F and B:T is logically impossible for $A =$ "$x > 5$" and $B =$ "$x > 7$".

A	B	$A \wedge B$
F	F	F
F	T	L/I
T	F	F
T	T	T

A	B	$A \wedge B$
F	F	F
T	F	F
T	T	T

□

Remark 3.2. It was mentioned in the definition of logically impossible T/F combinations that there had to be some relationship between the inputs, i.e., some placeholders or some fixed but unspecified constants to make this an interesting definitions.

Consider what happens if you have two statements A and B for which this is not the case. For example, let $A :=$ "All tomatoes are blue" (obviously false) and $B :=$ "Arkansas is a state of the U.S.A." (obviously true).

For those two specific statements we know upfront that we have A:F and B:T, so why bother with the other three cases? In other words, the appropriate truth table is either of those two:

A	B	$A \wedge B$
F	F	L/I
F	T	F
T	F	L/I
T	T	L/I

A	B	$A \wedge B$
F	T	F

□

Remark 3.3. We chose for a more compact notation to place "L/I" into one of the statement columns but be aware that the L/I attribute really belongs to certain combinations of the T/F values of the inputs. In other words,

the L/I attribute belongs to certain rows of the truth table. A more accurate way would be to place L/I into a separate status column and place "N/A" or "-" or nothing into all columns other than those for the inputs:

Status	A	B	$A \wedge B$
	F	F	F
L/I	F	T	-
	T	F	F
	T	T	T

□

Of course more than two input statements can be involved when discussing logical impossibility. The following example will show this.

[16] e.g., if we have a statement function $P : x \mapsto P(x)$ and we look at the statements $P(x_0)$ for which x_0 belongs to the UoD of P or a certain subset thereof

Example 3.12. Let U, V, W, Z be the statement functions

$U := x \mapsto U(x) :=$ "$x \in [0, 4]$",
$V := x \mapsto V(x) :=$ "$x \notin \emptyset$",
$W := x \mapsto W(x) :=$ "$x < -1$",
$Z := x \mapsto Z(x) :=$ "$x > 2$"

with UoD $\mathbb{R}$ in each case. Let Q be a statement function that is built from U, V, W, Z with the help of logical operators.

We observe the following:

- **a.** $V(x)$ is always true because the empty set does not contain any elements.
- **a′.** In other words, there is no x in the UoD for which $V(x)$ is false.
- **b.** There is no x in the UoD for which $W(x)$ and $Z(x)$ can both be true.

The following rows in the resulting truth table yield an L/I regardless whether we enter a truth value of T or F into anyone of the "•" entries.

$U(x)$	$V(x)$	$W(y)$	$Z(x)$	$Q(x)$
•	F	•	•	L/I
•	•	T	T	L/I

□

Remark 3.4. As in example 3.12 above let

$U := U(x) :=$ "$x \in [0, 4]$", $V := V(x) :=$ "$x \notin \emptyset$", $W := W(x) :=$ "$x < -1$", $Z := Z(x) :=$ "$x > 2$".

a. The statement [17] $Q(x) := \neg(U(x) \wedge V(x)) \wedge W(x) \wedge Z(x)$ can never be true, regardless of x.

To see this directly note again that $V(x)$ is trivially true for any x because the emptyset by definition does not contain any elements. It follows that $U(x) \wedge V(x)$ means "$x \in [0, 4]$" and $Q(x)$ means "$x < 0$" $\wedge$ "$x > 4$" $\wedge$ "$x < -1$" $\wedge$ "$x > 2$" which is equivalent to "$x < -1$" $\wedge$ "$x > 4$" and certainly false for any x in the UoD, i.e., $x \in \mathbb{R}$.

Alternatively we can use the results from example 3.12 where we found out that $W(x)$ and $Z(x)$ cannot both be true at the same time.

The remaining rows in the resulting truth table yield an F for $Q(x)$ regardless of the truth values of $U(x)$ and $V(x)$ because $W(x) \wedge Z(x)$ is false, hence $Q(x) =$ whatever $\wedge\big(W(x) \wedge Z(x)\big)$ is false for those remaining rows.

$U(x)$	$V(x)$	$W(y)$	$Z(x)$	$Q(x)$
•	•	F	F	F
•	•	F	T	F
•	•	T	F	F

b. Let $R : x \mapsto R(x) := \neg Q(x)$ be the statement function with UoD $\mathbb{R}$ which represents for each x in the UoD the opposite of Q. Because $Q(x)$ is false for all x, $R(x)$ is true for all x in the universe of discourse for x. □

Statements which are true or false under all circumstances like the statements $R(x)$ and $Q(x)$ from the remark above deserve special names.

Definition 3.10 (Tautologies and contradictions)**.** A **tautology** is a statement which is true under all circumstances, i.e., under all combinations of truth values which are not logically impossible.

[17] It is tough to come up with some decent examples of compound statements if the only operators at your disposal so far are negation and conjunction.

A **contradiction** is a statement which is false under all circumstances.

We write T_0 for the tautology "$1 = 1$" and F_0 for the contradiction "$1 = 0$". This gives us a convenient way to incorporate statements which are true or false under all circumstances into formulas that build compound statements. $\square$

Example 3.13. Here are some examples of tautologies.

a. The statements $R(x)$ of remark 3.4 are tautologies.

b. T_0 is a boring example of a tautology. So is any true statement without any variables such as "$9 + 12 = 21$" and "a cat is not a cow".

c. There are formulas involving arbitrary statements which are tautologies. We will show that for any two statements A and B the statement $P := \neg(A \wedge \neg A)$ is a tautology.

Here are some examples of contradictions.

d. The statements $Q(x)$ of remark 3.4 are contradictions.

e. F_0 is a boring example of a contradiction. So is any false statement without any variables such as "$9 + 12 = 50$" and "a dog is a whale".

f. There are formulas involving arbitrary statements which are contradictions. We will show that for any two statements A and B the statement $Q := (A \wedge \neg A) \wedge B$ is a contradiction. $\square$

*Proof of **c** and **f**:*

$P = \neg(A \wedge \neg A)$ (last column) has entries all T, hence P is a tautology.
$Q = (A \wedge \neg A) \wedge B$ (next to last column) has entries all F, hence Q is a contradiction.

A	B	$\neg A$	$A \wedge \neg A$	$(A \wedge \neg A) \wedge B$	$\neg(A \wedge \neg A)$
F	*F*	*T*	*F*	*F*	*T*
F	*T*	*T*	*F*	*F*	*T*
T	*F*	*F*	*F*	*F*	*T*
T	*T*	*F*	*F*	*F*	*T*

■

We now continue with the conjunction operator.

Example 3.14. In the following let x, y be two (fixed but arbitrary) integers and let $A(x) :=$ "$x \in \mathbb{N}$" and $B(y) :=$ "$y \in \mathbb{Z}$ **and** $y > 0$". Be sure to understand that $A(x)$ and $B(y)$ are in fact statements and not predicates, because the symbols x, y are bound from the start and hence cannot be considered variables of the predicates $A :=$ "$x \in \mathbb{N}$" and $B :=$ "$y \in \mathbb{Z}$ **and** $y > 0$".

We will reuse the statements $A(x)$ and $B(y)$ in examples for the subsequently defined logical operators.

a. If no assumptions are made about a relationship between x and y then all four T/F combinations are possible and, to explore conjunction, we must deal with the full truth table

$A(x)$	$B(y)$	$A(x) \wedge B(y)$
F	F	F
F	T	F
T	F	F
T	T	T

b. On the other hand, if $x < y$ then the truth of $A(x)$ implies that of $B(y)$ because if y is an integer which dominates some natural number x then we have $y > x \geqq 1 > 0$, i.e., y is an integer bigger than zero, i.e., truth of $A(x)$ and falseness of $B(y)$ are incompatible.

It follows that the combination T/F is L/I. We discard the corresponding row and restrict ourselves to the truth table

$A(x)$	$B(y)$	$A(x) \wedge B(y)$
F	F	F
F	T	F
T	T	T

c. Even better, if $x = y$, i.e., we compare truth/falsehood of $A(x)$ with that of $B(x)$, we only need to worry about the two combinations F/F and T/T for the following reason: The set of positive integers is the set $\{1, 2, \dots\}$ and this is, by definition, the set $\mathbb{N}$ of all natural numbers. This means that the statements "$x \in \mathbb{N}$" and "$y \in \mathbb{Z}$ **and** $y > 0$" are just two different ways of expressing the same thing.

It follows that either both $A(x)$ and $B(x)$ are true or both are false. We discard the logically impossible combinations F/T and T/F and restrict ourselves to the truth table

$A(x)$	$B(x)$	$A(x) \wedge B(x)$
F	F	F
T	T	T

□

3.3.3 Biconditional and Logical Equivalence Operators – Part 1

Definition 3.11 (Double arrow operator (biconditional))**.** The **double arrow operator** [18] is represented by the symbol "$\leftrightarrow$" and read "A double arrow B". $A \leftrightarrow B$ is **true** if and only if either both A and B are **true or** both A and B are **false**.

(3.9) Truth table for $A \leftrightarrow B$:

A	B	$A \leftrightarrow B$
F	F	T
F	T	F
T	F	F
T	T	T

□

Definition 3.12 (Logical equivalence operator)**.** Two statements A and B are **logically equivalent** if the statement $A \leftrightarrow B$ is a tautology, i.e., if the combinations A:**true**, B:**false** and A:**false**, B:**true** both are logically impossible.

We write $A \Leftrightarrow B$ and we say "A if and only if B" to indicate that A and B are logically equivalent.

(3.10) Truth table for $A \Leftrightarrow B$:

A	B	$A \Leftrightarrow B$
F	F	T
F	T	L/I
T	F	L/I
T	T	T

□

[18] [4] Bryant, Kirby Course Notes for MAD 2104 calls this operator the **equivalence operator** but we abstain from that terminology because "A is equivalent to B" has a different meaning and is written $A \Leftrightarrow B$.

The discussion of the $\leftrightarrow$ and $\Leftrightarrow$ operators will be continued in ch.3.3.6 (Biconditional and Logical Equivalence Operators – Part 2) on p.42

3.3.4 Inclusive and Exclusive Or

Definition 3.13 (Disjunction)**.** The **disjunction operator** is represented by the symbols "$\vee$" or "**or**". The expression A **or** B is **true** if and only if either A or B is **true**.

(3.11) Truth table for $A \vee B$:

A	B	$A \vee B$
F	F	F
F	T	T
T	F	T
T	T	T

The **or** connective generalizes to more than two statements $A_1, A_2, \dots, A_n$ in the obvious manner:

$A_1 \vee A_2 \vee \cdots \vee A_n$ is **true** if and only if at least one of $A_1, A_2, \dots, A_n$ is **true** and **false** otherwise, i.e., if each of the A_k is **false**. □

Example 3.15. As in example 3.14 let $x, y \in \mathbb{Z}$ and let
$A(x) :=$ "$x \in \mathbb{N}$" and $B(y) :=$ "$y \in \mathbb{Z}$ **and** $y > 0$"

a. If no assumptions are made about a relationship between x and y then all four T/F combinations are possible and, to explore conjunction, we must deal with the full truth table

$A(x)$	$B(y)$	$A(x) \vee B(y)$
F	F	F
F	T	T
T	F	T
T	T	T

b. Let $x < y$. We have seen in example 3.14.**b** that the combination T/F is impossible and we can restrict ourselves to the simplified truth table

$A(x)$	$B(y)$	$A(x) \vee B(y)$
F	F	F
F	T	T
T	T	T

c. Now let $x = y$. We have seen in example 3.14.**c** that either both $A(x)$ and $B(y) = B(x)$ are true or both are false. Because the combinations F/T and T/F are impossible we can restrict ourselves to the simplified truth table

$A(x)$	$B(x)$	$A(x) \wedge B(x)$
F	F	F
T	T	T

□

Definition 3.14 (Exclusive or)**.** The **exclusive or operator** is represented by the symbol "**xor**".[19] A **xor** B is **true** if and only if either A or B is **true** (but not both as is the case for the inclusive or).

(3.12) Truth table for A **xor** B:

A	B	A **xor** B
F	F	F
F	T	T
T	F	T
T	T	F

[19] Some documents such as [4] Bryant, Kirby Course Notes for MAD 2104. also use the symbol $\oplus$.

□

Example 3.16. As in example 3.14 let $x, y \in \mathbb{Z}$ and let
$A(x) :=$ "$x \in \mathbb{N}$" and $B(y) :=$ "$y \in \mathbb{Z}$ **and** $y > 0$"

a. If no assumptions are made about a relationship between x and y then all four T/F combinations are possible and, to explore conjunction, we must deal with the full truth table

$A(x)$	$B(y)$	$A(x)$ **xor** $B(y)$
F	F	F
F	T	T
T	F	T
T	T	F

b. Let $x < y$. We have seen in example 3.14.**b** that the combination T/F is impossible and we can restrict ourselves to the simplified truth table

$A(x)$	$B(y)$	$A(x)$ **xor** $B(y)$
F	F	F
F	T	T
T	T	F

c. Now let $x = y$. We have seen in example 3.14.**c** that either both $A(x)$ and $B(y) = B(x)$ are true or both are false. Because the combinations F/T and T/F are impossible we can restrict ourselves to the simplified truth table

$A(x)$	$B(x)$	$A(x)$ **xor** $B(x)$
F	F	F
T	T	F

This last truth table is remarkable. The truth values for $A(x)$ **xor** $B(x)$ are **false** in each row, hence it is a contradiction as defined in def.3.10 on p.32. □

Remark 3.5. Note that the truth values for $A \leftrightarrow B$ are the exact opposites of those for A **xor** B:

$A \leftrightarrow B$ is true exactly when both A and B have the same truth value whereas A **xor** B is true exactly when A and B have opposite truth values. In other words,

$A \leftrightarrow B$ is true whenever $\neg[A$ **xor** $B]$ is true and false whenever $\neg[A$ **xor** $B]$ is false. □

Exercise 3.1. use that last remark to prove that for any two statements A and B the compound statement

$$[A \leftrightarrow B] \leftrightarrow \neg[A \textbf{ xor } B]$$

is a tautology. □

3.3.5 Arrow and Implication Operators

Definition 3.15 (Arrow operator)**.** The **arrow operator** [20] is represented by the symbol "$\rightarrow$". We read $A \rightarrow B$ as "A arrow B" but see remark 3.7 below for the interpretation "if A then B".

(3.13) Truth table for $A \rightarrow B$:

A	B	$A \rightarrow B$
F	F	T
F	T	T
T	F	F
T	T	T

[20] [4] Bryant, Kirby Course Notes for MAD 2104 calls this operator the **implication operator** but we abstain from that terminology because "A implies B" has a different meaning and is written $A \Rightarrow B$.

In other words, $A \to B$ is **false** if and only if A is **true** and B is **false**. □

Definition 3.16 (Implication operator)**.** We say that A **implies** B and we write

$$A \;\Rightarrow\; B \tag{3.14}$$

for two statements A and B if the statement $A \to B$ is a tautology, i.e., if the combination A: **true**, B: **false** is logically impossible.

(3.15) Truth table for $A \Rightarrow B$:

A	B	$A \Rightarrow B$
F	F	T
F	T	T
T	F	L/I
T	T	T

□

Remark 3.6. There are several ways to express $A \Rightarrow B$ in plain english:

Short form:	Interpret this as:
A implies B	The truth of A implies the truth of B
if A then B	if A is true then B is true
A only if B	A is true only if B is true
B if A	B is true if A is true
B whenever A	B is true whenever A is true
A is sufficient for B	The truth of A is sufficient for the truth of B
B is necessary for A	The truth of B is necessary for the truth of A

□

Theorem 3.1 (Transitivity of "$\Rightarrow$")**.** *Let A, B, C be three statements such that $A \Rightarrow B$ and $B \Rightarrow C$. Then $A \Rightarrow C$.*

Proof:

*$A \Rightarrow B$ means that the combination A:T, B:F is logically impossible because otherwise $A \to B$ would have a truth value of F and we would not have a tautology. Hence we can drop row 5 from the truth table on the right. Similarly we can drop row 7 because it contains the combination B:T, C:F which contradicts our assumption that $B \Rightarrow C$. But those are the only rows for which $A \to C$ yields **false** because only they contain the combination A:T, C:F. It follows that $A \to C$ is a tautology, i.e., $A \Rightarrow C$.*

	A	B	C
1	F	F	F
2	F	F	T
3	F	T	F
4	F	T	T
5	T	F	F
6	T	F	T
7	T	T	F
8	T	T	T

■

Theorem 3.2 (Transitivity of "$\to$")**.** *Let A, B, C be three statements.*

Then $\big[(A \to B) \wedge (B \to C)\big] \;\Rightarrow\; (A \to C)$.

Proof: We must show that $\big[(A \to B) \wedge (B \to C)\big] \;\to\; (A \to C)$ is a tautology. We do this by brute force and compute the truth table.

A	B	C	$A \to B$	$B \to C$	$P :=$ $(A \to B) \wedge (B \to C)$	$A \to C$	$P \to (A \to C)$
F	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	T	T	T	T	T	T	T
T	F	F	F	T	F	F	T
T	F	T	F	T	F	T	T
T	T	F	T	F	F	F	T
T	T	T	T	T	T	T	T

We see that the last column with the truth values for $\big[(A \to B) \wedge (B \to C)\big] \ \to \ (A \to C)$ *contains* ***true*** *everywhere and we have proved that this statement is a tautology.* ■

Definition 3.17. In the context of $A \to B$ and $A \Rightarrow B$ we call A the **premise** or the **hypothesis** [21] and we call B the **conclusion**. [22]

We call $B \to A$ the **converse** of $A \to B$ and we call $\neg B \to \neg A$ the **contrapositive** of $A \to B$.

We call $B \Rightarrow A$ the **converse** of $A \Rightarrow B$ and we call $\neg B \Rightarrow \neg A$ the **contrapositive** of $A \Rightarrow B$. □

Remark 3.7.

- **a.** The difference between $A \to B$ and $A \Rightarrow B$ is that $A \Rightarrow B$ implies a relation between the premise A and the conclusion B which renders the T/F combination A:T, B:F logically impossible, i.e., the pared down truth table has only **true** entries in the $A \Rightarrow B$ column. In other words, $A \Rightarrow B$ is the statement $A \to B$ in case the latter is a tautology as defined in def.3.10 on p.32.
- **b.** Both $A \to B$ and $A \Rightarrow B$ are interpreted as "if A then B" but we prefer in general to say "A arrow B" for $A \to B$ because outside the realm of logic $A \Rightarrow B$ is what mathematicians use when they refer to "If ... then " constructs to state and prove theorems.

□

Example 3.17. The converse of "if x is a dog then x is a mammal" is "if x is a mammal then x is a dog". You see that, regardless whether you look at it in the context of $\to$ or $\Rightarrow$, a "**if** . . . **then**" statement can be true whereas its converse will be false and vice versa.

The contrapositive of "if x is a dog then x is a mammal" is "if x is not a mammal then x is a not a dog". Switching to the contrapositive did not switch the truth value of the "**if** . . . **then**" statement. This is not an accident: see the Contrapositive Law (3.41) on p.46. □

Remark 3.8. What is the connection between the truth tables for $A \to B$, $A \Rightarrow B$ and modeling "if A then B"?

We answer this question as follows:

a. If the premise A is guaranteed to be false, you should be allowed to conclude from it anything you like:

[21] also called the **antecedent**

[22] Another word for conclusion is **consequent** .

Consider the following statements which are obviously false:

F_1 : "The average weight of a 30 year old person is 7 ounces",
F_2 : "The number 12.7 is an integer",
F_3 : "There are two odd integers m and n such that $m+n$ is odd",
F_4 : "All continuous functions are differentiable" [23]

and some that are known to be true:

T_1 : "The moon orbits the earth",
T_2 : "The number 12.7 is not an integer",
T_3 : "If m and n are even integers then $m+n$ is even",
T_4 : "All differentiable functions are continuous"

a1. What about the statement "**if** F_3 **then** T_1": "If There are two odd integers m and n such that $m+n$ is odd then the moon orbits the earth"? This may not make a lot of sense to you, but consider this:

The truth of "**if** F_3 **then** T_1" **is not the same as** the truth of just F_1. No absolute claim is made that the moon orbits the earth. You are only asked to concede such is the case under the assumption that two odd integers can be found whose sum is odd. But we know that no such integers exist, i.e., we are dealing with a vacuous premise and there is no obligation on our part to show that the moon indeed orbits the earth! Because of this we should have no problem to accept the validity of "**if** F_3 **then** T_1". Keep in mind though that knowing that **if** F_3 **then** T_1 will not help to establish the truth or falseness of T_1!

a2. Now what about the statement "**if** F_3 **then** F_2": "If There are two odd integers m and n such that $m+n$ is odd then the number 12.7 is an integer"? The truth of this implication should be much easier to understand than allowing to conclude something false from something false:

When was the last time that someone bragged "Yesterday I did xyz" and you responded with something like "If you did xyz then I am the queen of Sheba" in the serene knowledge that there is no way that this person could have possibly done xyz? You know that you have no burden of proof to show that you are the queen of Sheba because you did not make this an absolute claim: You hedged that such is only the case if it is true that the other person in fact did xyz yesterday.

So, yes, the argument "**if** F_3 **then** F_2". sounds OK and we should accept it as true but, as in the case of "**if** F_3 **then** T_1". this has no bearing on the truth or falseness of F_2.

To summarize, "**if** F **then** B". should be true, no matter what you plug in for B. We thus have obtained the first two rows of a sensible truth table for A → B:

A	B	A → B
F	F	T
F	T	T

b. Is it OK to say that if the premise A is true then we may infer that the conclusion B is also true? Definitely! There is nothing wrong with "**if** T_2 **then** T_4", i.e., the statement "If The number 12.7 is not an integer then all differentiable functions are continuous"

[23] A counterexample is the function $f(x) = |x|$ because it is continuous everywhere but not differentiable at $x = 0$.

We can add the fourth row but we do not have #3 yet:

A	B	A → B
F	F	T
F	T	T
T	F	??
T	T	T

c. Is it OK to say that, if the premise A is true, we may say in parallel that A implies B even if the conclusion B is false? No way! Let's assume that Jane is a goldfish. Then A: "Jane is a fish" is true and B: "Jane is a rocket scientist" is false. It is definitely NOT OK to say, under those circumstances, "If Jane is a fish" then Jane is a rocket scientist". Contrast that with this modification that fits case b: "If Jane is a fish' then Jane is **not** a rocket scientist". No one should have a problem with that! We now can complete row #3: T → F is false.

We now have the complete truth table for $A \to B$ and it matches the one in def.3.15:

A	B	$A \to B$
F	F	T
F	T	T
T	F	F
T	T	T

The truth table (3.15) for $A \Rightarrow B$ is then derived from that for $A \to B$ by demanding that A and B be such that $A \to B$ cannot be false, i.e, the combination A:F, B:T must be logically impossible:

A	B	$A \Rightarrow B$
F	F	T
F	T	T
T	F	L/I
T	T	T

We arrived in this remark at the truth tables for $A \to B$ and $A \Rightarrow B$ based on what seems to be reasonable. But the discipline of logic is as exacting a subject as abstract math and the process had to be done in reverse: We first had to **define** $A \to B$ and $A \Rightarrow B$ by means of the truth tables given in def.3.15 and def.3.16 and from there we justified why these operators appropriately model "if A then B". □

Example 3.18. As in example 3.14 let $x, y \in \mathbb{Z}$ and let
$A(x) :=$ "$x \in \mathbb{N}$" and $B(y) :=$ "$y \in \mathbb{Z}$ **and** $y > 0$"

a. If no assumptions are made about a relationship between x and y then all four T/F combinations are possible and, to explore conjunction, we must deal with the full truth table

$A(x)$	$B(y)$	$A(x) \to B(y)$
F	F	T
F	T	T
T	F	F
T	T	T

b. Let $x < y$. We have seen in example 3.14.**b** that the combination T/F is impossible and we can restrict ourselves to the simplified truth table

$A(x)$	$B(y)$	$A(x) \to B(y)$
F	F	T
F	T	T
T	T	T

c. Now let $x = y$. We have seen in example 3.14.**c** that either both $A(x)$ and $B(y) = B(x)$ are true or both are false. Because the combinations F/T and T/F are impossible we can restrict ourselves to the simplified truth table

$A(x)$	$B(x)$	$A(x) \to B(x)$
F	F	T
T	T	T

We see that $A(x) \to B(y)$ is a tautology in case that $x < y$ or $x = y$. $\square$

We have seen that some work was involved to show that the "$A(x) \to B(y)$" statement of the last example is a tautology. How do we interpret this?

> *If you show that a "**if** P **then** Q" statement is a tautology then you have demonstrated that a true premise necessarily results in a true conclusion. You have "**proved**" the validity of the conclusion Q from the validity of the hypothesis P.*

*The next example is a modification of the previous one. We replace the statements $A(x)$ and $B(y)$ with statement functions $x \mapsto A(x), y \mapsto B(y), (x,y) \mapsto C(x,y)$. and replace $A(x) \to B(y)$ with an equivalent $\to$ statement which involves those three statement functions. Our goal is now to show that this new **if** ... **then** statement is a tautology for all x and y which belong to their universes of discourse.*

Example 3.19. Let $\mathscr{U}_x := \mathscr{U}_y := \mathbb{Z}$ be the UoDs for the variables x and y.

Let $A : \mathscr{U}_x \to \mathscr{S}$ with $x \mapsto$ "$x \in \mathbb{N}$",
$B : \mathscr{U}_y \to \mathscr{S}$ with $y \mapsto$ "$y \in \mathbb{Z}$ **and** $y > 0$",
$C : \mathscr{U}_x \times \mathscr{U}_y \to \mathscr{S}$ with $(x,y) \mapsto$ "$x < y$".

Let us try to show that for any x in the UoD of x and y in the UoD of y, i.e., for any two integers x and y, the function value $T(x,y)$ of the statement function

(3.16) $\quad T : \mathscr{U}_x \times \mathscr{U}_y \to \mathscr{S}$ with $(x,y) \mapsto T(x,y) := \big[\big(A(x) \land C(x,y)\big) \ \to \ B(y)\big]$ is a tautology.

Note that

a. The last arrow in (3.16) is the arrow operator $\to$, not the function assignment operator $\mapsto$.
b. if we can demonstrate that (3.16) is correct then we can replace $\big(A(x) \land C(x,y)\big) \ \to \ B(y)$ with $\big(A(x) \land C(x,y)\big) \ \Rightarrow \ B(y)$. We interpret this as having proved the (trivial) Theorem: It is true for all integers x and y that if $x \in \mathbb{N}$ and $x < y$ then $y \in \mathbb{Z}$ **and** $y > 0$.

The trick is of course to think of x and y not as placeholders but as fixed but unspecified integers. Then $A(x), B(y)$ and $C(x,y)$ are ordinary statements and we can build truth tables just as always. Observe that we now have three "inputs" $A(x), B(y)$ and $C(x,y)$ and the full truth table contains nine entries.

We need not worry about numbers x and y whose combination (x,y) results in the falseness of the premise $A(x) \land C(x,y)$ because **false** $\to$ $B(y)$ always results in **true**. In other words we do not worry about any combination of x and y for which at least one of $A(x)$, $C(x,y)$ is false. To phrase it differently we focus on such x and y for which we have that both $A(x)$, $C(x,y)$ are true and eliminate all other rows from the truth table. There are only two cases to consider: either $B(y)$ is **false** or $B(y)$ is **true**:

$A(x)$	$C(x,y)$	$B(y)$	$A(x) \land C(x,y)$	$\big(A(x) \land C(x,y)\big) \to B(y)$
T	T	F	T	F
T	T	T	T	T

The proof is done if it can be shown that the first row is a logically impossible. We now look at the components $A(x)$, $C(x, y)$, $B(y)$ in context. We have seen in example 3.14**b.** that the assumed truth of $C(x, y)$ together with that of $A(x)$ is incompatible with $B(y)$ being false. This eliminates the first row from that last truth table and what remains is

$A(x)$	$C(x,y)$	$B(y)$	$A(x) \wedge C(x,y)$	$\big(A(x) \wedge C(x,y)\big) \to B(y)$
T	T	T	T	T

In other words we obtain the value **true** for all non-contradictory combinations in the last column of the truth table and this proves (3.16). ☐

Remark 3.9. Let us compare example 3.18.**b** with example 3.19. Besides using statements in the former and predicates in the latter a more subtle difference is that, because x and y were assumed to be known from the outset,

example 3.18.**b** allowed us to formulate a truth table in which none of the statements had to explicitly refer to the condition $x < y$.

In contrast to this we had to introduce in example 3.19 the predicate $C =$ "$x < y$" to bring this condition into the truth tables

Was there any advantage of switching from statements to predicates and adding a significant amount of complexity in doing so? The answer is yes but it will only become clear when we introduce quantifiers for statement functions. ☐

We will come back to the subject of proofs in chapter 3.7.1 (Building blocks of mathematical theories) on p.57.

3.3.6 Biconditional and Logical Equivalence Operators – Part 2 (Understand this!)

This chapter continues the discussion of the $\leftrightarrow$ and $\Leftrightarrow$ operators from ch.3.3.3 (Biconditional and Logical Equivalence Operators – Part 1) on p.34.

Remark 3.10.

a. Equivalence $A \Leftrightarrow B$ provides a "**replacement principle for statements**": Logically equivalent statements are not "semantically identical" but they cannot be distinguished as far as their "logic content", i.e., the circumstances under which they are true or false are concerned.

b. Note that $A \Leftrightarrow B$ means the same as the following: A is true whenever B is true and A is false whenever B is false because this is the same as saying that, in a truth table that contains entries for A and B, each row either has the value T in both columns or the value F in both columns. This in turn is the same as saying that the column for $A \leftrightarrow B$ has T in each row, i.e., $A \leftrightarrow B$ is a tautology.

b′. There is not much value to **b** if A and B are simple statements but things become a lot more interesting if compound statements like $A := \neg(P \wedge Q)$ and $B := \neg P \vee \neg Q$ are looked at. ☐

We illustrate the above remark with the following theorem.

Theorem 3.3 (De Morgan's laws for statements)**.** *Let A and B be statements. Then we have the following* *logical equivalences:*

$$\neg(A \land B) \Leftrightarrow \neg A \lor \neg B, \tag{3.17}$$

$$\neg(A \lor B) \Leftrightarrow \neg A \land \neg B. \tag{3.18}$$

Those formulas generalize to n statements $A_1, A_2, \dots, A_n$ as follows:

$$\neg(A_1 \land A_2 \land \cdots \land A_n) \Leftrightarrow \neg A_1 \lor \neg A_2 \lor \cdots \lor \neg A_n, \tag{3.19}$$

$$\neg(A_1 \lor A_2 \lor \cdots \lor A_n) \Leftrightarrow \neg A_1 \land \neg A_2 \land \cdots \land \neg A_n. \tag{3.20}$$

Proof of 3.17: Here is the truth table for both $\neg(A \land B)$ and $\neg A \lor \neg B$ depending on the truth values of A and B.

A	B	$A \land B$	$\neg(A \land B)$	$\neg A$	$\neg B$	$\neg A \lor \neg B$	$[\neg(A \land B)] \leftrightarrow [\neg A \lor \neg B]$
F	*F*	*F*	*T*	*T*	*T*	*T*	*T*
F	*T*	*F*	*T*	*T*	*F*	*T*	*T*
T	*F*	*F*	*T*	*F*	*T*	*T*	*T*
T	*T*	*T*	*F*	*F*	*F*	*F*	*T*

This proves the validity of 3.17. Note that the last column of the truth table is superfluous because getting T in each row follows from the fact that the rows of the statement to the left and the one to the right of "$\leftrightarrow$" both contain the same entries T-T-T-F. The column has been included because it illustrates what was said in remark 3.10.

Proof of 3.18: Left as an exercise. ■

Example 3.20. As in example 3.14 let $x, y \in \mathbb{Z}$ and let
$A(x) :=$ "$x \in \mathbb{N}$" and $B(y) :=$ "$y \in \mathbb{Z}$ **and** $y > 0$"

a. If no assumptions are made about a relationship between x and y then the full truth table needs all four entries and we obtain

$A(x)$	$B(y)$	$A(x) \leftrightarrow B(y)$
F	F	T
F	T	F
T	F	F
T	T	T

b. Let $x < y$. We have seen in example 3.14 that the combination T/F is impossible and we can restrict ourselves to the simplified truth table

$A(x)$	$B(y)$	$A(x) \rightarrow B(y)$
F	F	T
F	T	F
T	T	T

c. Now let $x = y$. We have seen in example 3.14.**c** that then either $A(x)$ and $B(y) = B(x)$ must both be true or they must both be false. Because the combinations F/T and T/F are impossible we can restrict ourselves to the simplified truth table

$A(x)$	$B(x)$	$A(x) \leftrightarrow B(x)$
F	F	T
T	T	T

It follows that for any given number x the statement $A(x) \leftrightarrow B(x)$ is always true, irrespective of the truth values of $A(x)$ and $B(x)$. Hence $A(x) \leftrightarrow B(x)$ is a tautology and we can write $A(x) \Leftrightarrow B(x)$ for all x. □

3.3.7 More Examples of Tautologies and Contradictions (Understand this!)

Now that we have all logical operators at our disposal we can give additional examples of tautologies and contradictions.

Example 3.21. In the following let P, Q, R be three arbitrary statements, let x, y be two (fixed but arbitrary) integers and let $A(x) :=$ "$x \in \mathbb{N}$" and $B(y) :=$ "$y \in \mathbb{Z}$ **and** $y > 0$". (see example 3.14 on p. 33).

a. Tautologies:

T_0,
$A_1 :=$ "$5 + 7 = 12$",
$A_2 :=$ "Any integer is even **or** odd",
$A_3 := P \lor \neg P$ (Tertium non datur or law of the excluded middle),
$A_4 := P \lor T_0$,
$A_5 := (P \land Q) \lor (P \land \neg Q)$,
$A_6 := (P \to Q) \leftrightarrow (\neg P \lor Q)$ (Implication is logically equivalent to an **or** statement),
$A_7 := [$"$x < y$"$ \land A(x)] \to B(y)$ (see 3.18.**b** on p.40),
$A_8 := A(x) \leftrightarrow B(x)$ (see 3.18.**c**).

Note that we can express the fact that A_6, A_7, A_8 are tautologies as follows:

$$(P \to Q) \Leftrightarrow (\neg P \lor Q), \quad [\text{“}x < y\text{”} \land A(x)] \Rightarrow B(y), \quad A(x) \Leftrightarrow B(x).$$

b. Contradictions:

F_0,
$B_1 :=$ "$5 + 7 = 15$",
$B_2 :=$ "There are some non-zero numbers x such that $x = 2x$",
$B_3 := P \land \neg P$,
$B_3 := P \land F_0$,
$B_4 := F_0 \land (P \lor \neg P)$,
$B_5 := [\neg P \lor \neg Q] \land [P \land Q]$,
$B_6 := A(x)$ **xor** $B(x)$ (see 3.16.**c** on p. 36). □

Proof that A_3 is a tautology:

P	$\neg P$	$P \lor \neg P$
F	*T*	*T*
T	*F*	*T*

Proof that A_4 is a tautology:

P	T_0	$P \vee T_0$
F	T	T
T	T	T

Note that even though there are two inputs, P and T_0, there are only two valid combinations of truth values because the only choice for T_0 is ***true****.*

Proof that A_6 is a tautology:

P	Q	$P \to Q$	$\neg P$	$\neg P \vee Q$	$(P \to Q) \leftrightarrow (\neg P \vee Q)$
F	F	T	T	T	T
F	T	T	T	T	T
T	F	F	F	F	T
T	T	T	F	T	T

■

Remark 3.11. The interesting tautologies and contradictions are not those involving only specific statements such as $T_0, F_0, A_1, A_2, B_1, B_2$, from above but those statements like A_5, A_6, B_4 and B_5 which specify formulas relating the general statements P, Q and R. □

3.4 Statement Equivalences (Understand this!)

Symbolic logic has a collection of very useful statement equivalences which are given here. They were taken from ch.2 on logic, subchapter 2.4 (Important Logical Equivalences) of [4] Bryant, Kirby Course Notes for MAD 2104.

Theorem 3.4. *Let P, Q, R be statements.*

a. *Identity Laws:*

$$P \wedge T_0 \Leftrightarrow P \tag{3.21}$$
$$P \vee F_0 \Leftrightarrow P \tag{3.22}$$

b. *Domination Laws:*

$$P \vee T_0 \Leftrightarrow T_0 \tag{3.23}$$
$$P \wedge F_0 \Leftrightarrow F_0 \tag{3.24}$$

c. *Idempotent Laws:*

$$P \vee P \Leftrightarrow P \tag{3.25}$$
$$P \wedge P \Leftrightarrow P \tag{3.26}$$

d. *Double Negation Law:*

$$\neg(\neg P) \Leftrightarrow P \tag{3.27}$$

e. *Commutative Laws:*

$$P \lor Q \Leftrightarrow Q \lor P \tag{3.28}$$
$$P \land Q \Leftrightarrow Q \land P \tag{3.29}$$

f. *Associative Laws:*

$$(P \lor Q) \lor R \Leftrightarrow P \lor (Q \lor R) \tag{3.30}$$
$$\textit{hence} \ (P \lor Q) \lor R \Leftrightarrow P \lor Q \lor R \tag{3.31}$$
$$(P \land Q) \land R \Leftrightarrow P \land (Q \land R) \tag{3.32}$$
$$\textit{hence} \ (P \land Q) \land R \Leftrightarrow P \land Q \land R \tag{3.33}$$

g. *Distributive Laws:*

$$P \lor (Q \land R) \Leftrightarrow (P \lor Q) \land (P \lor R) \tag{3.34}$$
$$P \land (Q \lor R) \Leftrightarrow (P \land Q) \lor (P \land R) \tag{3.35}$$

h. *De Morgan's Laws:* [24]

$$\neg(P \land Q) \Leftrightarrow \neg P \lor \neg Q \tag{3.36}$$
$$\neg(P \lor Q) \Leftrightarrow \neg P \land \neg Q \tag{3.37}$$

i. *Absorption Laws:*

$$P \land (P \lor Q) \Leftrightarrow P \tag{3.38}$$
$$P \lor (P \land Q) \Leftrightarrow P \tag{3.39}$$

j. *Implication Law:*

$$(P \rightarrow Q) \Leftrightarrow (\neg P \lor Q) \tag{3.40}$$

You should remember this formula because the fact that implication can be expressed as an OR statement is often extremely useful when showing that two statements are logically equivalent.

k. *Contrapositive Laws:*

$$(P \rightarrow Q) \Leftrightarrow (\neg Q \rightarrow \neg P) \tag{3.41}$$
$$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P) \tag{3.42}$$

l. *Tautology:*

$$(P \lor \neg P) \Leftrightarrow T_0 \tag{3.43}$$

m. *Contradiction:*

$$(P \land \neg P) \Leftrightarrow F_0 \tag{3.44}$$

[24] This is theorem 3.3 (De Morgan's laws for statements).

n. *Equivalence:* (3.45) $$(P \to Q) \wedge (Q \to P) \Leftrightarrow (P \leftrightarrow Q)$$

The proof for only some of the laws stated above are given here. You can prove all others by writing out the truth tables to show that left and right sides of the $\dots \Leftrightarrow \dots$ *statements are indeed logically equivalent.*

*Proof of **h** (De Morgan's laws):*
See theorem 3.3 on p.42.

*Proof of **j** (implication law):*
We prove (3.40) using a truth table:

We see that the entries T-T-F-T in the $\neg P \vee Q$ *column match those given for* $P \to Q$ *in def.3.15 on p.36 of the arrow operator. This proves the logical equivalence of those statements.*

P	Q	$\neg P$	$\neg P \vee Q$
F	F	T	T
F	T	T	T
T	F	F	F
T	T	F	T

*Proof of **k** (contrapositive law for* $\to$*):*
*We prove (3.41) with the help of the previously given laws **a** through **j**:*

$$(P \to Q) \overset{(j)}{\Leftrightarrow} (\neg P \vee Q) \overset{(e)}{\Leftrightarrow} (Q \vee \neg P) \overset{(d)}{\Leftrightarrow} (\neg(\neg Q) \vee \neg P) \overset{(j)}{\Leftrightarrow} (\neg Q \to \neg P)$$

■

Example 3.22. Use the logical equivalences of thm.3.4 to prove that $\neg\big(\neg A \wedge (A \wedge B)\big)$ is a tautology.
□

Solution:

	$\neg\big(\neg A \wedge (A \wedge B)\big)$	
$\Leftrightarrow$	$\neg(\neg A) \vee \neg(A \wedge B)$	*De Morgan's Law* (3.36)
$\Leftrightarrow$	$A \vee (\neg A \vee \neg B)$	*De Morgan* (3.36) + *Double negation* (3.27)
$\Leftrightarrow$	$(A \vee \neg A) \vee \neg B$	*Associative law* (3.30)
$\Leftrightarrow$	$T_0 \vee \neg B$	*Tautology* (3.43)
$\Leftrightarrow$	T_0	*Commutative Law* (3.28) + *Domination Law* (3.23) ■

Example 3.23. Find a simple expression for the negation of the statement "**if** you come before 6:00 **then** I'll take you to the movies". □

Solution: Let $A :=$ *"You come before 6:00" and* $B :=$ *"I'll take you to the movies". Our task is to find a simple logical equivalent to* $\neg(A \to B)$*. We proceed as follows:*

$$\neg(A \to B) \overset{(j)}{\Leftrightarrow} \neg(\neg A \vee B) \overset{(h)}{\Leftrightarrow} (\neg(\neg A) \wedge \neg B) \overset{(d)}{\Leftrightarrow} (A \wedge \neg B)$$

*This translates into the statement "you come before 6:00 **and** I won't take you to the movies".*

■

Remark 3.12. Now that we accept that such logical expressions are DEFINED by their truth tables, we must accept the following: if two logical expressions with two statements A and B as input have the same truth table, then they are logically equivalent and we may interchangeably use one or the other in a proof. □

3.5 The Connection Between Formulas for Statements and for Sets (Understand this!)

Given statements a, b and sets A, B you may have the impression that there are connections between $a \wedge b$ and $A \cap B$, between $a \vee b$ and $A \cup B$, between $\neg a$ and $A^{\complement}$, etc. We will briefly explore this.

In this chapter we switch to small letters for statements and statement functions and use capital letters to denote sets. You have already seen an example in the introduction.

We assume the existence of a universal set $\mathscr{U}$ of which all sets are subsets.

All statements will be of the form $a(x) =$ "$x \in A$" for some set $A \subset \mathscr{U}$. In other words we associate with such a set A the following statement function:

$$a : \mathscr{U} \to \mathscr{S}, \qquad x \mapsto a(x) =: \text{“}x \in A\text{''} \tag{3.46}$$

This relationship establishes a correspondence between the subset A of $\mathscr{U}$ and the predicate $a =$ "$x \in A$" with UoD $\mathscr{U}$. We write $a \cong A$ for this correspondence.

Example 3.24. Let $a \cong A$ and $b \cong B$.

We have

a. $T_0 \cong \mathscr{U}$, $F_0 \cong \emptyset$

b. $\neg a : x \mapsto \neg a(x) = \neg$"$x \in A$" evaluates to a true statement if and only if $x \notin A$, i.e. $x \in A^{\complement}$. Hence $\neg a \cong A^{\complement}$.

c. $a \wedge b : x \mapsto a(x) \wedge b(x) =$ "$x \in A$ **and** $x \in B$" evaluates to a true statement if and only if $x \in A \cap B$. Hence $a \wedge b \cong A \cap B$.

d. $a \vee b : x \mapsto a(x) \vee b(x) =$ "$x \in A$ **or** $x \in B$" evaluates to a true statement if and only if $x \in A \cup B$. Hence $a \vee b \cong A \cup B$. □

We expand the table of formulas for statements given in thm 3.4 on p.45 of ch.3.4 (Statement equivalences) with a third column which shows the corresponding relation for sets. Having a translation of statement relations to set relations allows you to use Venn diagrams as a visualization aid.

Theorem 3.5. *For a set $\mathscr{U}$ Let p, q, r be statement functions and let $P, Q, R \subseteq \mathscr{U}$ such that $p \cong P$, $q \cong Q$, $r \cong R$. Then we have the following:*

***a.** Identity:*	(3.47)	$p \wedge T_0 \Leftrightarrow p$	$P \cap \mathscr{U} = P$
	(3.48)	$p \vee F_0 \Leftrightarrow p$	$P \cup \emptyset = P$
***b.** Domination:*	(3.49)	$p \vee T_0 \Leftrightarrow T_0$	$P \cup \mathscr{U} = \mathscr{U}$
	(3.50)	$p \wedge F_0 \Leftrightarrow F_0$	$P \cap \emptyset = \emptyset$
***c.** Idempotency:*	(3.51)	$p \vee p \Leftrightarrow p$	$P \cup P = P$
	(3.52)	$p \wedge p \Leftrightarrow p$	$P \cap P = P$
***d.** Double Negation:*	(3.53)	$\neg(\neg p) \Leftrightarrow p$	$(P^{\complement})^{\complement} = P$
***e.** Commutative:*	(3.54)	$p \vee q \Leftrightarrow q \vee p$	$P \cup Q = Q \cup P$
	(3.55)	$p \wedge q \Leftrightarrow q \wedge p$	$P \cap Q = Q \cap P$
***f.** Associative:*	(3.56)	$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	$(P \cup Q) \cup R = P \cup (Q \cup R)$
	(3.57)	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	$(P \cap Q) \cap R = P \cap (Q \cap R)$
***g.** Distributive:*	(3.58)	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	$P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$
	(3.59)	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	$P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)$
***h.** De Morgan:*	(3.60)	$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$	$(P \cap Q)^{\complement} = P^{\complement} \cup Q^{\complement}$
	(3.61)	$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	$(P \cup Q)^{\complement} = P^{\complement} \cap Q^{\complement}$
***i.** Absorption:*	(3.62)	$p \wedge (p \vee q) \Leftrightarrow p$	$P \cap (P \cup Q) = P$
	(3.63)	$p \vee (p \wedge q) \Leftrightarrow p$	$P \cup (P \cap Q) = P$

j1. *Implication 1:*

(3.64) $(p \to q) \Leftrightarrow (\neg p \vee q)$ $\qquad (P \setminus Q)^{\complement} = P^{\complement} \cup Q$

Interpretation: $p(x) \to q(x)$*, i.e., "*$x \in P$*" → "*$x \in Q$*" is* ***true*** *if and only if* $p(x)$*:T,* $q(x)$*:F is L/I., i.e., if and only if* $x \notin P \cap Q^{\complement} = P \setminus Q$*, i.e.,* $x \in (P \setminus Q)^{\complement}$*.*

(3.65) $p \Rightarrow q$

j2. *Implication 2:*

$P \setminus Q = \emptyset$*, i.e.,* $P \subseteq Q$

Note that we are not dealing with $p \to q$ *but with* $p \Rightarrow q$ *where we assume for all* x *a relation between* p *and* q *which renders* $p(x)$*:T,* $q(x)$*:F logically impossible.*

k. *Contrapositive:*

(3.66) $(P \to Q) \Leftrightarrow (\neg Q \to \neg P)$ $\qquad P^{\complement} \cup Q = Q \cup P^{\complement}$

(3.67) $(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$ $\qquad P \subseteq Q \Leftrightarrow Q^{\complement} \subseteq P^{\complement}$

l. *Tautology:*

(3.68) $(P \vee \neg P) \Leftrightarrow T_0$ $\qquad P \cup P^{\complement} = \mathscr{U}$

m. *Contradiction:*

(3.69) $(P \wedge \neg P) \Leftrightarrow F_0$ $\qquad P \cap P^{\complement} = \emptyset$

n1. *Equivalence 1:*

(3.70) $(p \to q) \wedge (q \to p) \Leftrightarrow (p \leftrightarrow q)$ $\qquad (P^{\complement} \cup Q) \cap (Q^{\complement} \cup P) = \{x : x$ *both in* P, Q ***or*** x *neither in* P *nor in* Q $\}$

n2. *Equivalence 2:*

(3.71) $(p \Rightarrow q) \wedge (q \Rightarrow p) \Leftrightarrow (p \Leftrightarrow q)$ $\qquad (P \subseteq Q)$ ***and*** $(Q \subseteq P) \Leftrightarrow (P = Q)$

Proof: The set equalities are evident except for the following:

Proof of Equivalence 1:

$$
\begin{aligned}
(P^{\complement} \cup Q) \cap (Q^{\complement} \cup P) &= \left[(P^{\complement} \cup Q) \cap Q^{\complement}\right] \cup \left[(P^{\complement} \cup Q) \cap P\right] \\
&= (P^{\complement} \cap Q^{\complement}) \cup (Q \cap Q^{\complement}) \cup (P^{\complement} \cap P) \cup (Q \cap P) \\
&= (P^{\complement} \cap Q^{\complement}) \cup (Q \cap P) \\
&= \{x : x \textit{ neither in } P \textit{ nor in } Q \textbf{ or } x \textit{ both in } P, Q\}.
\end{aligned}
$$

■

3.6 Quantifiers for Statement Functions

This chapter has been kept rather brief. You can find more about quantifiers in ch.2 on logic, subchapter ch.2.3 (Predicates and Quantifiers) of [4] Bryant, Kirby Course Notes for MAD 2104.

3.6.1 Quantifiers for One–Variable Statement Functions

Definition 3.18 (Quantifiers)**.** Let $A : \mathscr{U} \to \mathscr{S},\ x \mapsto A(x)$ be a statement function of a single variable x with UoD $\mathscr{U}$ for x.

a. The **universal quantification** of the predicate A is the statement

(3.72) $$\text{“For all } x\ A(x)\text{''}, \quad \text{written } \forall x A(x).$$

The above is a short for "$A(x)$ is true for each $x \in \mathscr{U}$". We call the symbol $\forall$ the **universal quantifier** symbol.

b. The **existential quantification** of the predicate A is the statement

(3.73) $$\text{“For some } x\ A(x)\text{''}, \quad \text{written } \exists x A(x).$$

The above is a short for "There exists $x \in \mathscr{U}$ such that $A(x)$ is true". [25] We call the symbol $\exists$ the **existential quantifier** symbol.

c. The **unique existential quantification** of the predicate A is the statement

(3.74) $$\text{“There exists unique } x \text{ such that } A(x)\text{''}, \quad \text{written } \exists! x A(x).$$

The above is a short for "There exists a unique $x \in \mathscr{U}$ such that $A(x)$ is true". [26] We call the symbol $\exists!$ the **unique existential quantifier** symbol. □

Example 3.25. Let $A : [-3, 3] \to \mathscr{S}$ be the statement function $x \mapsto$ "$x^2 - 4 = 0$".

Let $C := \forall x A(x)$ $D := \exists x A(x)$ and $E := \exists! x A(x)$. Then

$C =$ "for all $x \in [-3, 3]$ it is true that $x^2 - 4 = 0$"

[25] Equivalently, "$A(x)$ is true for some $x \in \mathscr{U}$" or "$A(x)$ is true for at least one $x \in \mathscr{U}$".
[26] Equivalently, "$A(x)$ is true for exactly one $x \in \mathscr{U}$".

$D =$ "there is at least one $x \in [-3, 3]$ such that $x^2 - 4 = 0$"

$E =$ "there is exactly one $x \in [-3, 3]$ such that $x^2 - 4 = 0$"

Note that each of C, D, E is in fact a statement because each one is either true or false: Clearly the zeroes of the function $f(x) = x^2 - 4$ in the interval $-3 \leqq x \leqq 3$ are $x = \pm 2$. It follows that D is a true statement and A and C are false statements. $\square$

Example 3.26. Let $\mathscr{U} := \{$ all human beings $\}$ be the UoD for the following three predicates:

$S(x) :=$ "x is a student at NYU",
$C(x) :=$ "x cheats when taking tests",
$H(x) :=$ "x is honest",

Let us translate the following three english verbiage statements into formulas:

$A_1 :=$ "All humans are NYU students",
$A_2 :=$ "All NYU students cheat on tests",
$A_3 :=$ "Any NYU student who cheats on tests is not honest".

Solution:

$A_1 = \forall x\ S(x)$,
$A_2 = \forall x\ [S(x) \to C(x)]$,
$A_3 = \forall x\ [(S(x) \land C(x)) \to \neg H(x)]$. $\square$

Example 3.27. We continue example 3.26.

Let us simplify $A_3 = \forall x\ [(S(x) \land C(x)) \to \neg H(x)]$.

It is clear that "$A(x)$ is true for all x" is equivalent to "There is no x such that $A(x)$ is false". In other words, we have for any statement function A the following:

$$\forall x\ A(x) \ \Leftrightarrow\ \neg\big[\,\exists x\ (\neg A(x))\,\big].$$

But A_3 is the form $\forall x\ A(x)$: replace $A(x)$ with $(S(x) \land C(x)) \to \neg H(x)$.

It follows that

$$A_3 \ \Leftrightarrow\ \neg\big[\,\exists x\ (\neg\big(S(x) \land C(x)) \to \neg H(x)\big))\,\big].$$

What a mess! let us drop the "(x)" everywhere and the above becomes

$$A_3 \ \Leftrightarrow\ \neg\big[\,\exists x\ (\neg\big(S \land C) \to \neg H\big))\,\big].$$

We have seen in example 3.23 on p.47 that for any two statements P and Q the equivalence $\neg(P \to Q) \ \Leftrightarrow\ (P \land \neg Q)$ is true.

Let us apply this with $P := S \land C$ and $Q := \neg H$. We obtain

$$A_3 \ \Leftrightarrow \neg\big[\,\exists x\ \big((S \land C) \land \neg(\neg H)\big)\,\big].\ \Leftrightarrow\ \neg\big[\,\exists x\ \big(S \land C \land H\big)\,\big].$$

where we obtained the last equivalence by applying the double negation law to $\neg(\neg H)$ and the associative law for $\land$ to remove the parentheses from $(S \land C) \land H$.

As a last step we bring back the "(x)" terms and obtain

$$A_3 \Leftrightarrow \neg\exists x\, [S(x) \wedge C(x) \wedge H(x)].$$

In other words, A_3 means "There is no one who is an NYU student and who cheats on tests and is honest". This should make sense if you remember the original meaning of A_3: "Any NYU student who cheats on tests is not honest". $\square$

3.6.2 Quantifiers for Two–Variable Statement Functions

We now discuss quantifiers for statement functions of two variables. Things become a lot more interesting because we can mix up $\forall$, $\exists$ and $\exists!$.

Unless mentioned otherwise B denotes the statement function of two variables

(3.75) $$B : \mathscr{U}_x \times \mathscr{U}_y \to \mathscr{S}, \quad x \mapsto B(x,y)$$

It follows that the unverses of discourse are $\mathscr{U}_x$ for x and $\mathscr{U}_y$ for y.

We need a quantifier for each variable to bind the expression $B(x,y)$ with placeholders x and y into a statement, i.e., into something that will be true or false. This done by example as follows:

Definition 3.19 (Doubly quantified expressions)**.** Here is a table of statements involving two quantifiers and their meanings.

a.	$\forall x \forall y B(x,y)$	"for all $x \in \mathscr{U}_x$ and for all $y \in \mathscr{U}_y$ (we have the truth of) $B(x,y)$",
b.	$\forall x \exists y B(x,y)$	"for all $x \in \mathscr{U}_x$ there exists (at least one) $y \in \mathscr{U}_y$ such that $B(x,y)$",
c.	$\exists x \forall y B(x,y)$	"there exists (at least one) $x \in \mathscr{U}_x$ such that for all $y \in \mathscr{U}_y$ $B(x,y)$",
d.	$\exists! x \forall y B(x,y)$	"there exists exactly one $x \in \mathscr{U}_x$ such that for all $y \in \mathscr{U}_y$ $B(x,y)$",
e.	$\exists x \exists y B(x,y)$	"there exists (at least one) $x \in \mathscr{U}_x$ and (at least one) $y \in \mathscr{U}_y$ such that $B(x,y)$". $\square$

Example 3.28. Let $\mathscr{U}_x := \mathbb{N}, \mathscr{U}_y := \mathbb{Z}$ and $B : \mathscr{U}_x \times \mathscr{U}_y \to \mathscr{S}, \quad (x,y) \mapsto B(x,y) :=$ "$x + y = 1$". Then

a.	$\forall x \forall y B(x,y)$	**false**
b.	$\forall x \exists y B(x,y)$	**true**: for the given x choose $y := 1 - x$.
c.	$\exists y \forall x B(x,y)$	**false**
d.	$\forall y \exists x B(x,y)$	**false**: If you choose $y > 0$ then the only x that satisfies the equation $x + y = 1$ is $x = 1 - y \leq 0$, i.e., $x \notin \mathbb{N}$, the UoD for x.
e.	$\exists! x \forall y B(x,y)$	**false**
f.	$\exists x \exists y B(x,y)$	**true**: choose $x := 10$ and $y := -9$.

Understand the different outcomes of **b**, **c** and **d** and remember this:

a.	The order in which the qualifiers are applied is important. $\forall x \exists y$ generally does not mean the same as $\exists y \forall x$.
b.	Interchanging variable names in the qualifiers is not OK. $\forall x \exists y$ generally does not mean the same as $\forall y \exists x$.

Proposition 3.1. *Note the following:*

(3.76) $$\forall x \forall y B(x,y) \Leftrightarrow \forall y \forall x B(x,y)$$

(3.77) $$\exists x \exists y B(x,y) \Leftrightarrow \exists y \exists x B(x,y)$$

(3.78) $$\forall x \exists y B(x,y) \not\Leftrightarrow \exists y \forall x B(x,y)$$

(3.79) $$\exists y \forall x B(x,y) \Rightarrow \forall x \exists y B(x,y)$$

Proof: *(3.76) and (3.77) follow from* ***a*** *and* ***e*** *in def. 3.19 and we saw an example for (3.76) in the previous example.*

The last item is not so obvious. We argue as follows: Assume that $\exists y \forall x B(x,y)$ *is true. Then there is some* $y_0 \in \mathscr{U}_y$ *such that* $B(x, y_0$ *is true for all* $x \in \mathscr{U}_x$.

Why does that imply the truth of $\forall x \exists y B(x,y)$*, i.e., for all* $x \in \mathscr{U}_x$ *you can pick some* $y \in \mathscr{U}_y$ *such that* $B(x,y)$ *is true? Here is the answer: Pick* y_0*. This works because, by assumption,* $B(x, y_0)$ *is true for all* $x \in \mathscr{U}_x$. ■

Remark 3.13. The last part of the proof of (3.79) is worth a closer look:

"$\forall x \exists y \ldots$" only tells you that for all x there will be some y which generally depends on x, something we sometimes emphasize using "functional notation" $y = y(x)$.

"$\exists y \forall x \ldots$" does more: it postulates the existence of some y_0 which is suitable for each x in its UoD. The assignment $y(x) = y_0$ is constant in x! □

Remark 3.14 (Partially quantified statement functions)**.** Given a statement function

$$B : \mathscr{U}_x \times \mathscr{U}_y \to \mathscr{S}, \quad x \mapsto B(x,y)$$

with two place holders x and y, we can elect to use only one quantifier for either x or y. If we only quantify x then we only bind x and y still remains a placeholder and if we only quantify y then we only bind y and x still remains a placeholder. □

Example 3.29. Let $\mathscr{U}_x$:= { all students at this party } and $\mathscr{U}_y$:= { "Linear Algebra", Discrete Mathematics", "Multivariable Calculus", "Ordinary Differential Equations", "Complex Variables", "Graph Theory", "Real Analysis" }.

Let A := "x studies y" be the two-variable statement function with UoD $\mathscr{U}_x$ for x and UoD $\mathscr{U}_y$ for y, i.e.,

$A : \mathscr{U}_x \times \mathscr{U}_y \to \mathscr{S}, \quad (x,y) \mapsto A(x,y) =$ "x studies y".

Then $B := \forall x\ A(x,y)$ is the one-variable predicate

$B : \mathscr{U}_y \to \mathscr{S}, \quad y \mapsto B(y) =$ "all students at this party study y"

and $C := \exists! y\ A(x,y)$ is the one-variable predicate

$C : \mathscr{U}_x \to \mathscr{S}, \quad x \mapsto C(x) =$ "x studies exactly one of the courses listed in $\mathscr{U}_y$". □

3.6.3 Quantifiers for Statement Functions of more than Two Variables

Remark 3.15. Although this document limits its scope to statement functions of one or two variables (see the note before remark 3.6 in ch.3.2 (Statements and statement functions)) we discuss briefly the use of quantifiers for predicates

$$A : \mathscr{U}_1 \times \mathscr{U}_2 \times \cdots \times \mathscr{U}_n \to \mathscr{S}, \quad (x_1, x_2, \dots, x_n) \mapsto A(x_1, x_2, \dots, x_n).$$

with n place holders.

Each one of those variables needs to be bound by one of the quantifiers $\forall, \exists, \exists!$ in order to obtain a statement, i.e., something that is either true or false. □

Example 3.30 (Continuity vs uniform continuity)**.** This example demonstrates the effect of switching a $\forall$ quantifier with an $\exists$ quantifier for a predicate with four variables. You will learn later that one quantification corresponds to ordinary continuity and the other corresponds to uniform continuity of a function. Do not worry if you do not understand how this example relates to continuity. The only point of interest here is the use of the quantifiers.

Let $a < b$ be two real numbers and let $f :]a, b[\to \mathbb{R}$ be a function which maps each x in its domain $]a, b[$ to a real number $y = f(x)$.

Let $\mathscr{U}_\varepsilon := \mathscr{U}_\delta :=]0, \infty[$ and $\mathscr{U}_x := \mathscr{U}_{x'} :=]a, b[$. Let $P : \mathscr{U}_x \times \mathscr{U}_{x'} \times \mathscr{U}_\delta \times \mathscr{U}_\varepsilon \to \mathscr{S}$ be the predicate

$$(x, x', \delta, \varepsilon) \mapsto P(x, x', \delta, \varepsilon) \; := \; \text{“if } |x - x'| < \delta \text{ then } |f(x) - f(x')| < \varepsilon\text{“}.$$

Let $A := \forall \varepsilon \; \forall x \; \exists \delta \; \forall x' P(x, x', \delta, \varepsilon)$ Then A being true is equivalent to saying that the function f is continuous at each point $x \in]a, b[$. [27]

Let $B := \forall \varepsilon \; \exists \delta \; \forall x \; \forall x' P(x, x', \delta, \varepsilon)$. Then B being true is equivalent to saying that the function f is uniformly continuous in $]a, b[$. [28]

The difference between A and B is that in statement A the variable δ whose existence is required may depend on both ε and x, i.e., $\delta = \delta(\varepsilon, x$

On the other hand, to satisfy B, a δ must be found which still may depend on ε but it must be suitable for <u>all</u> $x \in]a, b[$, i.e., $\delta = \delta(\varepsilon$. □

Remark 3.16 (Partially quantified statement functions)**.** What was said in remark 3.14 about partial qualification of two-variable predicates generalizes to more than two variables: If A is a statement function with n variables and we use quantifiers for only $m < n$ of those variables then $n - m$ variables in the resulting expression remain unbound and this expression becomes a statement function of those unbound variables.

For example, if $A(w, x, y, z)$ is a four-variable predicate then $B : (x, z) \mapsto \big[\forall y \; \neg \exists w \; A(w, x, y, z)\big]$ defines a two-variable predicate B which inherits the UoDs for x and z from the original statement function A. □

[27] See def.10.31 (ε-δ continuity) on p.210.
[28] See def.10.34 (Uniform continuity of functions) on p.218.

3.6.4 Quantifiers and Negation (Understand this!)

Negation of statements involving quantifiers is governed by

Theorem 3.6 (De Morgan's laws for quantifiers)**.** *Let A be a statement function with UoD $\mathscr{U}$. Then*

a.	$\neg\big(\forall x A(x)\big) \Leftrightarrow \exists x\, \neg A(x)$	*"It is **not** true that $A(x)$ is true for all x" $\Leftrightarrow$ "There is some x for which $A(x)$ is **not** true"*
b.	$\neg\big(\exists x A(x)\big) \Leftrightarrow \forall x\, \neg A(x)$	*"There is **no** x for which $A(x)$ is true" $\Leftrightarrow$ "$A(x)$ is **not** true for all x "*

*Proof of **a**: Not given here but you can find it in ch.2 on logic, subchapter 3.11 (De Morgan's Laws for Quantifiers) of [4] Bryant, Kirby Course Notes for MAD 2104.*

*Proof of **b**: Let $\mathscr{U}_x$ be the UoD for x.*

The truth of $\neg\big(\exists x A(x)\big)$ means that $\exists x A(x)$ is false, i.e., $A(x)$ is false for all $x \in \mathscr{U}_x$. This is equivalent to stating that $\neg A(x)$ is true for all $x \in \mathscr{U}_x$ and this is by definition, the truth of $\forall x\, \neg A(x)$. ■

You can use the formulas above for negation of statements of more than one variable with more than one quantifier using the following method, demonstrated here by example.

Example 3.31. Negate the statement $\exists x \forall y P(x,y)$, i.e., move the $\neg$ operator of $\neg\exists x \forall y P(x,y)$ to the right past all quantifiers.

The key is to introduce an intermittent predicate $A : x \mapsto A(x) := \big[\forall y P(x,y)\big]$. We obtain

$$\big[\neg\exists x \forall y P(x,y)\big] \Leftrightarrow \big[\neg\exists x A(x)\big] \overset{(\mathbf{b})}{\Leftrightarrow} \big[\forall x \neg A(x)\big] \Leftrightarrow \big[\forall x(\neg\forall y P(x,y))\big]$$
$$\overset{(\mathbf{a})}{\Leftrightarrow} \big[\forall x(\exists y \neg P(x,y))\big]. \quad \square$$

Example 3.32. As in example 3.31, negate the statement $\exists x \forall y P(x,y)$ but do so using parentheses instead of explicitly defining an intermittent predicate.

Here is the solution:

$$\big[\neg\exists x \forall y P(x,y)\big] \Leftrightarrow \big[\neg\exists x\big(\forall y P(x,y)\big)\big] \overset{(\mathbf{b})}{\Leftrightarrow} \big[\forall x \neg\big(\forall y P(x,y)\big)\big] \Leftrightarrow \big[\forall x(\neg\forall y P(x,y))\big]$$
$$\overset{(\mathbf{a})}{\Leftrightarrow} \big[\forall x(\exists y \neg P(x,y))\big]. \quad \square$$

3.7 Proofs (Understand this!)

*We have informally discussed proofs in examples 3.18 and 3.19 of chapter 3.3.5 (Arrow and Implication Operators) on p.36 and seen in two simple cases how a proof can be done by building a single truth table for an **if**... **then** statement and showing that it is a tautology. In this chapter we take a deeper look at the concept of "proof".*

Many subjects discussed here follow closely ch.3 (Methods of Proofs) of [4] Bryant, Kirby Course Notes for MAD 2104.

3.7.1 Building Blocks of Mathematical Theories

Some of the terminology definitions in notations 3.2 and 3.4 were taken almost literally from ch.3 (Methods of Proofs), subchapter 1 (Logical Arguments and Formal Proofs) of [4] Bryant, Kirby Course Notes for MAD 2104.

Notations 3.2 (Axioms, rules of inferences and assertions)**.**

- **a.** An **axiom** is a statement that is true by definition. No justification such as a proof needs to be given.
- **b.** A **rule of inference** is a logical rule that is used to deduce the truth of a statement from the truth of others.
- **c.** For some statements it is not clear whether they are true for false. Even if a statement is known to be true there might be someone like a student taking a test who is given the task to demonstrate, i.e., prove its truth. In this context we call a statement an **assertion** and we call it a **valid assertion** if it can be shown to be true. An assertion which is not known to be true by anyone is often called a **conjecture**. □

Example 3.33. Let $A :=$ "all continuous functions are differentiable" (known to be false [29]) and $B :=$ "all differentiable functions are continuous" (known to be true). A homework problem in calculus may ask the students to figure out which of the four statements $A, \neg A, B, \neg B$ are valid assertions and give proofs to that effect. □

Remark 3.17.

a. Goldbach's conjecture states that every even integer greater than 2 can be expressed as the sum of two primes, i.e., integers p greater than 1 which can be divided evenly by no natural number other than p ($p/p = 1$) or 1 ($p/1 = p$). Goldbach came up with this in 1742, more than 250 years ago. No one has been able until now to either prove the validity of this assertion or provide a counterexample to prove its falsehood.

b. Fermat's conjecture was that there are no four numbers $a, b, c, n \in \mathbb{N}$ such that $n > 2$ and $a^n + b^n = c^n$. [30] This was stated by Pierre de Fermat in 1637 who then claimed that he had a proof. Unfortunately he never got around to write it down. A successful proof was finally published in 1994 by Andrew Wiles. Accordingly, Fermat's conjecture was rechristened Fermat's Last Theorem. □

[29] see remark 3.8 on p.38 in ch.3.3.5 (Arrow and Implication Operators).
[30] We have an elementary counterexample for $n = 2$: $3^2 + 4^2 = 25 = 5^2$.

Notations 3.3 (Proofs)**.** A **proof** is the demonstration that an assertion is valid. This demonstration must be detailed enough so that a person with sufficient expert knowledge can understand that we do indeed have a statement which is true for all logically possible combinations of T/F values. To show that the arguments given in this demonstration are valid, available tools are

a. the rules of inference which wil be discussed in section 3.7.2 (Rules of Inference) on p.60
b. logical equivalences for statements (see ch.3.3.6 (Biconditional and Logical Equivalence Operators – Part 2) on p.ch.42).

> In almost all cases the assertion in question is of the form "**if** P **then** C". Proving it means showing that the statement $P \to C$ is a tautology, i.e., it can be replaced by the stronger $P \Rightarrow C$ statement. The proof then consists of the demonstration that the combination P: **true**, C: **false** can be ruled out as logically impossible. In other words, assuming P: **true**, i.e., the truth of the premise, it must be shown that C: **true**, i.e., the conclusion then also is necessarily true.

Usually a proof is broken down into several "sub-proofs" which can be proved separately and where some or all of those steps again will be broken down into several steps ... You can picture this as a hierarchical upside down tree with a single node at the top. At the most detailed level at the bottom we have the leaf nodes. The proof of the entire statement is represented by that top node. □

Notations 3.4 (Theorems, lemmata and corollaries)**.**

a. A **theorem** is an assertion that can be proved to be true using definitions, axioms, previously proven theorems, and rules of inference.
b. A **lemma** (plural: lemmata) is a theorem whose main importance is that it can used to prove other theorems.
c. A **corollary** is a theorem whose truth is a fairly easy consequence of another theorem. □

Remark 3.18 (Terminology is different outside logic)**.** The terminology given in the above definitions is specific to the subject of mathematical logic. In other branches of mathematics and hence outside this chapter 3 different meanings are attached to those terms:

Each one of **lemma**, **proposition**, **theorem**, **corollary** is a theorem as defined above in notations 3.2, i.e., a statement that can be proved to be true. We distinguish those terms by comparing them to propositions:

a. Theorems are considered more important than propositions.
b. The main purpose of a lemma is to serve as a tool to prove other propositions or theorems.
c. A corollary is a fairly easy consequence of some lemma, proposition, theorem or other corollary.

□

It was mentioned as a footnote to the definition of a statement (def. 3.3 on p.25) that what we call a statement, [4] Bryant, Kirby calls a proposition and that we deviate from that approach because mathematics outside logic uses "proposition" to denote a theorem of lesser importance.

Any mathematical theory must start out with a collection of undefined terms and axioms that specify certain properties of those undefined terms.

There is no way to build a theory without undefined terms because the following will happen if you try to define every term: You define T_2 in terms of T_1, then you define T_3 in terms of T_2, etc. Two possibilities:

1. *Each of $T_1, T_2, T_3, \ldots$ are different and you end up with an infinite sequence of definitions.*
2. *At least one of those terms is repeated and there will be a circular chain of definitions.*

Neither case is acceptable if you want to specify the foundations of a mathematical system.

Example 3.34. Here are a few important examples of mathematical systems and their ingredients.

a. In Euclid's geometry of the plane some of the undefined terms are "point", "line segment" and "line". The five Euclidean axioms specify certain properties which relate those undefined terms. You may have heard of the fifth axiom, Euclid's parallel postulate. It has been reproduced here with small alterations from Wikipedia's "Euclidean geometry" entry: [31]
(It is postulated that) "if a line segment falling on two line segments makes the interior angles on the same side less than two right angles, the two line segments, if produced indefinitely, meet on that side on which are the angles less than the two right angles".

b. In the so called Zermelo-Fraenkel set theory which serves as the foundation for most of the math that has been done in the last 100 years, the concept of a "set" and the relation "is an element of" ($\in$) are undefined terms.

c. Chapters 1 and 2 of [1] Beck/Geoghegan list several axioms which stipulate the existence of a nonempty set called $\mathbb{Z}$ whose elements are called "integers" which you can "add" and "multiply". Certain algebraic properties such as "$a+b=b+a$" and "$c \cdot (a+b) = (c \cdot a) + (c \cdot a)$" are given as true and so is the existence of an additive neutral unit "0" and a multiplicative neutral unit "1". Besides those algebraic properties the existence of a strict subset $\mathbb{N}$ called "positive integers" is assumed which has, among others, the property that any $z \in \mathbb{Z}$ either satisfies $z \in \mathbb{Z}$ or $-z \in \mathbb{Z}$ or $z = 0$. Finally there is the induction axiom which states that if you create the sequence 1, $1+1$, $(1+1)+1$, $\ldots$ then you capture all of $\mathbb{N}$. This axiom is the basis for the principle of proof by mathematical induction (see def.2.12 on p. 18). □

Once we have the undefined terms and axioms for a mathematical system, we can begin defining new terms and proving theorems (or lemmas, or corollaries) within the system.

Remark 3.19 (Axioms vs. definitions)**.** You can define anything you want but if you are not careful you may have a logical contradiction and the set of all items that satisfy that definition is empty. In contrast, axioms will postulate the existence of an item or an entire collection of items which satisfy all axioms. If the axioms contradict each other we have a theory which is inconsistent and

[31] https://en.wikipedia.org/wiki/Euclidean_geometry#Axioms

the only way to deal with it is to discard it and rework its foundations An example for this was set theory in its early stages. Anything that you could phrase as "Let A be the set which contains ..." was fair game to define a set. We saw in remark 2.2 (Russell's Antinomy) on p.11 that this lead to problems so serious that they caused some of the leading mathematicians of the time to revisit the foundations of mathematics. □

Example 3.35. For example you can define an oddandeven integer to be any $z \in \mathbb{Z}$ which satisfies that $z - 212$ is an even number and $z + 48$ is an odd number and you can prove great things for such z. The problem is of course that the set of all oddandeven integers is empty! We have a definition which is useless for all practical purposes, but no mathematical harm is done.

On the other hand, if you add as an additional axiom for $\mathbb{Z}$ in example 3.34.**c** that $\mathbb{Z}$ must contain one or more oddandeven integers then you are in a conundrum because <u>you postulated the existence</u> of a set $\mathbb{Z}$ which satisfies all axioms and the existence of such a set is logically impossible! □

3.7.2 Rules of Inference

Remark 3.20 (Most important rules of inference)**.** In Notations 3.2 on p.57 we described the term "rule of inference" as "a logical rule that is used to deduce the truth of a statement from the truth of others". The most important rules of inference are those that allow you to draw a conclusion of the form "if A is true then I am allowed to deduce the the truth of C." This basically amounts to having is a list of premises $A_1, A_2, \dots, A_n$ and a conclusion C such that

(3.80) $$\text{the compound statement } \left[A_1 \wedge A_2 \wedge \cdots \wedge A_n\right] \to C \text{ is a tautology.}$$

In other words, the column for the conclusion C in the truth table for this statement must have the value **true** for each combination of truth values which is not logically impossible.

Observe that the order of the premises does not matter because the **and** connective is commutative.
□

Theorem 3.7. *Let $P_1, P_2, \dots, P_n$ and C be statements. Then the statement $(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \to C$ is a tautology if and only if the following combination of truth values is logically impossible:*

(3.81) $$P_j \text{ is } \textbf{\textit{true}} \text{ for each } j = 1, 2, \dots, n \text{ and } C \text{ is } \textbf{\textit{false}}.$$

Proof:

*Let $P := (P_1 \wedge P_2 \wedge \cdots \wedge P_n)$. Then "$P_j$ is **true** for each $j = 1, 2, \dots, n$" means according to the definition of the $\wedge$ operator the same as the truth of P. Hence proving the theorem is equivalent to proving that the statement $P \to C$ is a tautology if and only if the combination of truth values*

(3.82) $$P \text{ is } \textbf{\textit{true}} \text{ and } C \text{ is } \textbf{\textit{false}} \text{ is logically impossible.}$$

*In other words, we must prove that $P \to C$ is a tautology if and only if the row with the combination P:T, C:F, i.e., row 3, is logically impossible and can be ignored. This is is obvious as row 3 is the only one for which $P \to C$ evaluates to **false**.*

	P	C	$P \to C$
1.	F	F	T
2.	F	T	T
3.	T	F	***false***
4.	T	T	T

■

Notations 3.5. Rules of inference are commonly written in the following form:

Your explanations go into this area	A_1 A_2 $\cdots$ A_n ——— $\therefore C$

Read "$\therefore$" as "therefore". The following, more compact notation can also be found:

$$\frac{A_1, A_2, \dots, A_n}{\therefore C}$$

Theorem 3.8 (The three most important inference rules)**.** *The following lists three inference rules, i.e., those arrow statements are indeed tautoloties:*

(3.83)	*Modus Ponens (Law of detachment - the mode that affirms the antecedent (the premise))*	A $A \to C$ ——— $\therefore C$
(3.84)	*Modus Tollens (The mode that Denies the consequent (the conclusion))*	$\neg C$ $A \to C$ ——— $\therefore \neg A$
(3.85)	*Hypothetical syllogism*	$A \to B$ $B \to C$ ——— $\therefore A \to C$

Here is the compact notation:

Modus Ponens	*Modus Tollens*	*Hypothetical syllogism*
$\dfrac{A,\ A \to C}{\therefore C}$	$\dfrac{\neg C,\ A \to C}{\therefore \neg A}$	$\dfrac{A \to B,\ B \to C}{\therefore A \to C}$

Note that the proof that the hypothetical syllogism is a tautology was given in thm.3.1 on p.37

Proof:

■

Example 3.36. Here are five more inference rules.

(3.86) Disjunction Introduction
$$\frac{A}{\therefore A \lor B}$$

(3.87) Conjunction elimination
$$\frac{A \land B}{\therefore A}$$

(3.88) Disjunctive syllogism
$$\frac{\begin{array}{l} A \lor B \\ \neg A \end{array}}{\therefore B}$$

(3.89) Conjunction introduction
$$\frac{\begin{array}{l} A \\ B \end{array}}{\therefore A \land B}$$

(3.90) Constructive dilemma
$$\frac{\begin{array}{l} (A \to B) \land (C \to D) \\ A \lor C \end{array}}{\therefore B \lor D}$$

Compact notation:

Disjunction Introduction	Conjunction elimination	Disjunctive syllogism
$\frac{A}{\therefore A \lor B}$	$\frac{A \land B}{\therefore A}$	$\frac{A \lor B, \neg A}{\therefore B}$
Conjunction introduction	Constructive dilemma	
$\frac{A, B}{\therefore A \land B}$	$\frac{(A \to B) \land (C \to D), A \lor C}{\therefore B \lor D}$ -	

□

None of the rules of inference that were given in this chapter involve quantifiers. You can find information about that topic in ch.2, section 1.6 (Rules of Inference for Quantifiers) of [4] Bryant, Kirby Course Notes for MAD 2104.

3.7.3 An Example of a Direct Proof

We illustrate in detail a mathematical proof by applying some the tools you have learned so far in this chapter on logic. For an example we will prove the theorem that each polynomial is differentiable. We define a polynomial as a function $f(x) = \sum_{j=0}^{n} c_j x^j$ *for some* $n = 0, 1, 2, \dots$*, i.e., for some* $n \in \mathbb{Z}_{\geqq 0}$ *and we write* $\mathscr{D}$ *for the set of all differentiable functions. We now can formulate our theorem.*

Theorem 3.9. *Given the statements*

$$\textbf{a:}\quad A := \text{“}(n \in \mathbb{Z}_{\geqq 0}) \wedge (c_0 \in \mathbb{R}) \wedge (c_1 \in \mathbb{R}) \wedge \cdots \wedge (c_n \in \mathbb{R}) \wedge \ \big(f(x) = \sum_{j=0}^{n} c_j x^j\big)\text{''},$$

$$\textbf{b:}\quad B := \text{“}f(x) \in \mathscr{D}\text{''},$$

the following is valid: $A \Rightarrow B$. [32]

Proof:

We first collect the necessary ingredients.

We define the following statements which serve as abbreviations so that the formulas we will build are reasonably compact.

$$\begin{array}{ll}
\textbf{a:} & Z_j := \text{“}j \in \mathbb{Z}_{\geqq 0}\text{''}, \\
\textbf{b:} & C_j := Z_j \wedge \text{“}c_j \in \mathbb{R}\text{''}, \\
\textbf{c:}^{33} & X_j := Z_j \wedge \text{“}x^j \in \mathscr{D}\text{''}, \\
\textbf{d:} & D_j := Z_j \wedge \text{“}c_j x^j \in \mathscr{D}\text{''}, \\
\textbf{e:} & E := Z_n \wedge \text{“}f(x) = \sum_{j=0}^{n} c_j x^j\text{''}, \\
\textbf{f:} & B := \text{“}f(x) \in \mathscr{D}\text{''} \ \textit{(repeated for convenient reference)}
\end{array}$$

We now can write our theorem as

$$(Z_n \wedge C_0 \wedge C_1 \wedge \cdots \wedge C_n \wedge E) \to B. \tag{3.91}$$

[32] Note here and for the other theorems the use of $A_2 \Rightarrow B_2$ instead of $A_2 \to B_2$: We assume that Thm-2 has been proved, i.e., $A_2 \to B_2$ is a tautology.

[33] The expression x^j in **c** and **d** denotes the function $x \mapsto x^j$.

We assume that the following three theorems were proved previously, hence we may use them without giving a proof.

Theorem Thm-1: If $p(x)$ is a power of x, i.e., $p(x) = x^n$ for some $n = 0, 1, 2, \dots$, then is $p(x)$ differentiable.

We rewrite Thm-1 as an implication which uses the statements above. Let

$$A_1 := Z_n \ \wedge \ \text{“}p(x) = x^n\text{''}, \quad B_1 := X_n.$$

Then Thm-1 states that $A_1 \Rightarrow B_1$. [34]

Theorem Thm-2: The product of a constant (real number) and a differentiable function is differentiable.

We rewrite Thm-2 as an implication. Let
$A_2 :=$ “$c \in \mathbb{R}$'' $\wedge$ “$h(x) \in \mathscr{D}$'' $\wedge$ “$g(x) = c \cdot h(x)$'',
$B_2 :=$ “$h(x) \in \mathscr{D}$'',
Then Thm-2 states that $A_2 \Rightarrow B_2$.

Theorem Thm-3: The sum of differentiable functions is differentiable

We rewrite Thm-3 as an implication. Let

$$A_3 := \text{“}Z_n \wedge \text{“}h_1(x) \in \mathscr{D}\text{''} \wedge \text{“}h_2(x) \in \mathscr{D}\text{''} \wedge \dots \wedge \text{“}h_n(x) \in \mathscr{D}\text{''} \wedge \ \text{“}g(x) = \sum_{j=0}^{n} h_j(x)\text{''},$$

$B_3 :=$ “$g(x) \in \mathscr{D}$'',
Then Thm-3 states that $A_3 \Rightarrow B_3$.

	Assertion	*Reason*
a:	$Z_0, Z_1, \dots Z_n$	*evident from $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$*
b:	$C_0, C_1, \dots C_n$	*part of the premise of $A \to B$ (see (3.91))*
c:	$Z_j \to X_j \ \ (j = 0, 1, \dots n)$	*Thm-1 with $n := j$*
d:	$X_j \ \ (j = 0, 1, \dots n)$	***c** and modus ponens*
e:	$(Z_j \wedge C_j \wedge X_j) \to D_j \ \ (j = 0, 1, \dots n)$	*Thm-2 with $c := c_j$ and $h(x) := x^j$*
f:	$D_j \ \ (j = 0, 1, \dots n)$	***e** and modus ponens*
g:	E	*part of the premise of $A \to B$*
h:	$(Z_n \wedge D_0 \wedge D_1 \wedge \dots \wedge D_n \wedge E) \to B$	***g** and Thm-3 with $h_j(x) := c_j x^j$ and $g(x) := f(x)$*
i:	B	***h** and modus ponens*

We have demonstrated that the truth of the premise A of our theorem implies that of its conclusion B and this proves the theorem. ■

Remark 3.21. Let us reflect on the steps involved in the proof above.

[34] As is the case for the theorem we want to prove, note here and for Thm-2 and Thm-3 below the use of $A_1 \Rightarrow B_1$ instead of $A_1 \to B_1$: Thm-1 has been proved already, i.e., we know that $A_1 \to B_1$ is a tautology.

a: Break down all statements involved – not only those in the theorem you want to prove but also in all theorems, axioms and definitions you reference – into reusable components and name those components with a symbol so that it is easier to understand what assertions you employ and how they lead to the truth of other assertions. Example: D_j references the component $Z_j \ \wedge$ "$c_j x^j \in \mathscr{D}$" (which itself references the component $Z_j =$ "$j \in \mathbb{Z}''_{\geqq 0}$).

b: Rewrite the theorem to be proved as an implication $A \Rightarrow B$.

c: Do the same for the three other theorems that we assumed as already having been proved.

The following is specific to our example but can be modified to other problems.

d: Start by using the premise A and the definition $\mathbb{Z}_{\geqq 0} := \{0, 1, 2, \dots\}$ to get the first two rows. Show that what you have implies the truth of the premise of Thm-1 and then use the modus ponens inference rule to deduce the truth of its conclusion X_j. This allows X_j to become an additional assertion.

e: Use that new assertion to obtain the truth of the premise of Thm-2 and then use again modus ponens to deduce the truth of its conclusion D_j. Now D_j becomes an additional assertion.

f: Use that new assertion to obtain the truth of the premise of Thm-3 and then use again modus ponens to deduce the truth of D_j. Now D_j becomes an additional assertion. □

3.7.4 Invalid Proofs Due to Faulty Arguments

Remark 3.22 (Fallacies in logical arguments)**.** People who are not very analytical often commit the following errors in their argumentation:

(3.92)	Affirming the Consequent (proving the wrong direction)	$P \to Q$ Q ——— $\therefore P$
(3.93)	Denying the Antecedent (indirect proof in the wrong direction)	$P \to Q$ $\neg P$ ——— $\therefore \neg Q$
(3.94)	Circular Reasoning	The argument incorporates use of the (not yet proven) conclusion

□

The reason that the above are fallacies stems from the fact that the above "rules of inferences" are not tautologies.

Example 3.37 (Fallacies in reasoning)**. a.** Affirming the Consequent:
"If you are a great mathematician then you can add $2+2$". It is true that you can add $2+2$. You conclude that you are a great mathematician.

b. Denying the Antecedent:
"If this animal is a cat then it can run quickly". This is not a cat. You conclude that this animal cannot run quickly.

c. Circular Reasoning: [35]
"If xy is divisible by 5 then x is divisible by 5 or y is divisible by 5".

The following incorrect proof uses the yet to be proven fact that the factors can be divided evenly by 5.

Proof:
If xy is divisible by 5 then $xy = 5k$ for some $k \in \mathbb{Z}$. But then $x = 5m$ or $y = 5n$ for some $m, n \in \mathbb{Z}$ (this is the spot where the conclusion was used). Hence x is divisible by 5 or y is divisible by 5. ■

3.8 Categorization of Proofs (Understand this!)

*There are different methods by which you can attempt to prove an "**if** ... **then**" statement $P \Rightarrow Q$. They are:*

a. *Trivial proof*
b. *Vacuous proof*
c. *Direct proof*
d. *Proof by contrapositive*
e. *Indirect proof (proof by contradiction)*
f. *Proof by cases*

3.8.1 Trivial Proofs

The underlying principle of a trivial proof is the following: If we know that the conclusion Q is true then any implication $P \Rightarrow Q$ is valid, regardless of the hypothesis P.

Example 3.38 (Trivial proof)**.** Prove that if it rains at least 60 days per year in Miami then $25 + 35 = 60$.

Proof: There is nothing to prove as it is known that $25 + 35 = 60$. It is irrelevant whether or not in rains (or snows, if you prefer) 60 days per year in Miami. ■

3.8.2 Vacuous Proofs

*The underlying principle of a vacuous proof is that a wrong premise allows you to conclude anything you want: Both P:F, Q:F and P:F, Q:T yield **true** for $P \to Q$.*

[35] This is example 1.8.3 in ch.3 (Methods of Proofs) of [4] Bryant, Kirby Course Notes for MAD 2104.

*For example, it was mentioned in remark 2.3 (Elements of the empty set and their properties) on p.11 that you can state anything you like about the elements of the empty set as there are none. The underlying principle of proving this kind of assertion is that of a vacuous proof. We prove here assertion **d** of that remark.*

Theorem 3.10. *Let A be any set. Then $\emptyset \subseteq A$.*

Proof:

According to the definition of $\subseteq$ we must prove that if $x \in \emptyset$ then $x \in A$.

So let $x \in \emptyset$. We stop right here: "$x \in \emptyset$" is a false statement regardless of the nature of x because the empty set, by definition, does not contain any elements. It follows that $x \in A$. ■

Remark 3.23. You may ask: But is it not equally true that if $x \in \emptyset$ then $x \notin A$? The answer to that is YES, it is equally true that $x \in A$? and $x \notin A$?, but so what? First you'll find me an x that belongs to the empty set and **only then** am I required to show you that it both does and does not belong to A! □

3.8.3 Direct Proofs

In a direct proof of $P \Rightarrow Q$ we assume the truth of the hypothesis P and then employ logical equivalences, including the rules of inference, to show the truth of Q.

We proved in chapter 3.7.3 (An example of a direct proof) on p.63 that each polynomial is differentiable (theorem 3.9). That was an example of a direct proof.

3.8.4 Proof by Contrapositive

A proof by contrapositive makes use of the logical equivalence $(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$ (see the contrapositive law (3.42) on p46). We give a direct proof of $\neg Q \Rightarrow \neg P$, i.e., we assume the falseness of Q and prove that then P must also be false. Here is an example.

Theorem 3.11. *Let A, B be two subsets of some universal set Ω such that $A \cap B^{\complement} = \emptyset$. Then $A \subseteq B$.*

Proof: We prove the contrapositive instead: If $A \nsubseteq B$ then $A \cap B^{\complement} \neq \emptyset$.

So let us assume $A \nsubseteq B$. This means that not every element of A also belongs to B. In other words, there exists some $x \in A$ such that $x \notin B$. But then $x \in A \setminus B = A \cap B^{\complement}$, i.e., $A \cap B^{\complement} \neq \emptyset$.

We have proved from the negated conclusion $A \nsubseteq B$ the negated premise $A \cap B^{\complement} \neq \emptyset$. ■

3.8.5 Proof by Contradiction (Indirect Proof)

A proofs by contradiction are a generalization of proofs by contrapositive. We assume that it is possible for the implication $P \Rightarrow Q$ that the premise P can be true and Q can be false at the same time and construct the assumption of the truth of $P \cap \neg Q$ a statement R such that both R and $\neg R$ must be true. Here is an example.

Theorem 3.12. *Let $A \subseteq \mathbb{Z}$ with the following properties:*

$$m, n \in A \ \Rightarrow \ m + n \in A, \tag{3.95}$$

$$m, n \in A \ \Rightarrow \ mn \in A, \tag{3.96}$$

$$0 \notin A, \tag{3.97}$$

$$\text{if } n \in \mathbb{Z} \text{ then either } n \in A \text{ or } -n \in A \text{ or } n = 0. \tag{3.98}$$

Then $1 \in A$.

Proof by contradiction: Assume that A is a set of integers with properties (3.95) – (3.98) but that $1 \notin A$. We will show that then $1 \in A$ must be true. This finishes the proof because it is impossible that both $1 \notin A$ and $1 \in A$ are true.

***a.** It follows from $1 \notin A$ and (3.98) and $1 \neq 0$ that $-1 \in A$.*
***b.** It now follows from (3.96) that $(-1) \cdot (-1) \in A$, i.e., $1 \in A$.*

We have reached our contradiction. ■

Remark 3.24. In this simple proof the statement R for which both R and $\neg R$ were shown to be true happens to be the conclusion $1 \in A$. This generally does not need to be the case. □

3.8.6 Proof by Cases

Sometimes an assumption P is too messy to take on in its entirety and it is easier to break it down into two or more cases $P_1, P_2, \dots, P_n$ each of which only covers part of P but such that $P_1 \vee P_2 \vee \cdots \vee P_n$ covers all of it, i.e., we assume

$$P_1 \vee P_2 \vee \cdots \vee P_n \ \Leftrightarrow \ P. \tag{3.99}$$

Proof by cases then rests on the following theorem:

Theorem 3.13. *Let $P, Q, P_1 \vee P_2 \vee \cdots \vee P_n$ be statements such that (3.99) is true. Then*

$$\big(P \Rightarrow Q\big) \ \Leftrightarrow \ \big[\, (P_1 \Rightarrow Q) \ \vee \ (P_2 \Rightarrow Q) \ \vee \ \dots \ (P_n \Rightarrow Q) \,\big]. \tag{3.100}$$

Proof (outline): You would do the proof by induction. Prove (3.100) first for $n = 2$ by expressing $A \to B$ as $\neg A \vee B$ and then building a truth table that compares $\big(\neg(P_1 \vee P_2)\big) \vee Q$ with $\neg P_1 \vee Q \vee \neg P_2 \vee Q$.

Then do the induction step in which (3.99) becomes $P_1 \vee P_2 \vee \cdots \vee P_{n+1} \ \Leftrightarrow \ P$ by setting $A := P_1 \vee P_2 \vee \cdots \vee P_n$ and this way reducing the proof of (3.100) for $n+1$ to that of 2 components. You make the validity of $\big(A \Rightarrow Q\big) \ \Leftrightarrow \ \big[\, (P_1 \Rightarrow Q) \ \vee \ (P_2 \Rightarrow Q) \ \vee \ \dots \ (P_n \Rightarrow Q) \,\big]$ the induction assumption. ■

Theorem 3.14. *Prove that for any $x \in \mathbb{R}$ such that $x \neq 5$ we have*

$$\frac{x}{x-5} > 0 \ \Rightarrow \ \big[\, (x < 0) \text{ or } (x > 5) \,\big]. \tag{3.101}$$

Proof: There are two cases for which $x/(x-5) > 0$:
either both $x > 0$ and $x - 5 > 0$ or both $x < 0$ and $x - 5 < 0$. We write

$P :=$ "$x/(x-5) > 0$", [36] *$P_1 := x > 0$ **and** $x - 5 > 0$, $P_2 := x < 0$ **and** $x - 5 < 0$. Then $P = P_1 \vee P_2$.*

***case 1.** P_1:*
Obviously $x > 0$ and $x - 5 > 0$ if and only if $x > 5$, so we have proved $P_1 \Rightarrow (x > 5)$.

***case 2.** P_2:*
Obviously $x < 0$ and $x - 5 < 0$ if and only if $x < 0$, so we have proved $P_2 \Rightarrow (x < 0)$.

We now conclude from $P = P_1 \vee P_2$ and theorem 3.13 the validity of (3.101). ■

4 Functions and Relations (Study this!)

4.1 Cartesian Products and Relations

Definition 4.1 (Cartesian Product of two sets)**.** The **cartesian product** of two sets A and B is

$$A \times B \; := \; \{(a,b) : a \in A, b \in B\},$$

i.e., it consists of all pairs (a, b) with $a \in A$ and $b \in B$.

Two elements (a_1, b_1) and (a_2, b_2) are called **equal** if and only if $a_1 = a_2$ and $b_1 = b_2$. In this case we write $(a_1, b_1) = (a_2, b_2)$.

It follows from this definition of equality that the pairs (a, b) and (b, a) are different unless $a = b$. In other words, the order of a and b is important. We express this by saying that the cartesian product consists of **ordered pairs**.

As a shorthand, we abbreviate $A^2 := A \times A$. □

Example 4.1 (Coordinates in the plane)**.** Here is the most important example of a cartesian product of two sets. Let $A = B = \mathbb{R}$. Then $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ is the set of pairs of real numbers. I am sure you are familiar with what those are: They are just points in the plane, expressed by their x– and y–coordinates.
Examples of such points are are: $(1, 0) \in \mathbb{R}^2$ (a point on the x–axis), $(0, 1) \in \mathbb{R}^2$ (a point on the y–axis), $(1.234, -\sqrt{2}) \in \mathbb{R}^2$.

You should understand why we do not allow two pairs to be equal if we flip the coordinates: Of course $(1, 0)$ and $(0, 1)$ are different points in the xy–plane! □

Remark 4.1 (Function graphs as subsets of cartesian products)**.** We gave the preliminary definition of a function in def.3.1, p.23 of ch.3.1 (Prologue: Notation for Functions). [37] A function $f : X \to Y$ $y = f(x)$ which assigns any $x \in X$ to a unique function value $f(x) \in Y$, e.g., $f(x) = x^2$, is characterized by its graph

$$\Gamma_f \; := \; \{\big(x, f(x)\big) : x \in X\}$$

[36] $P :=$ "$x/(x-5) > 0$ **and** $x \neq 5$" if you want to be a stickler for precision
[37] The precise definition of a function will be given in section 4.2 on p.72.

which is a subset of the cartesian product $X \times Y$. For example, if $X = [-2, 3]$ and $Y = [0, 10]$ then $\Gamma_f := \{(x, x^2) : -2 \leqq x \leqq 3\}$ is a subset of $[-2, 3] \times [0, 10]$ $\square$

Remark 4.2 (Empty cartesian product)**.** Note that $A \times B = \emptyset$ if and only if $A = \emptyset$ or $B = \emptyset$ or both are empty. $\square$

Definition 4.2 (Relation)**.** Let X and Y be two sets and $R \subseteq X \times Y$ a subset of their cartesian product $X \times Y$. We call R a **relation** on (X, Y). A relation on (X, X) is simply called a relation on X. If $(x, y) \in R$ we say that x **and** y **are related** and we usually write xRy instead of $(x, y) \in R$.

A relation on X is **reflexive** if xRx for all $x \in X$. It is **symmetric** if $x_1 R x_2$ implies $x_2 R x_1$ for all $x_1, x_2 \in X$. It is **transitive** if $x_1 R x_2$ and $x_2 R x_3$ implies $x_1 R x_3$ for all $x_1, x_2, x_3 \in X$. It is **antisymmetric** if $x_1 R x_2$ and $x_2 R x_1$ implies $x_1 = x_2$ for all $x_1, x_2 \in X$. $\square$

Here are some examples of relations.

Example 4.2 (Equality as a relation)**.** Given a set X let $R := \{(x, x) : x \in X\}$, i.e., xRy if and only if $x = y$. This defines a relation on X which is reflexive, symmetric and transitive. $\square$

Example 4.3 (Set inclusion as a relation)**.** Given a set X let $R := \{(A, B) : A, B \subseteq X \text{ and } A \subseteq B\}$, i.e., ARB if and only if $A \subseteq B$. This defines a relation which is reflexive, antisymmetric and transitive. $\square$

Example 4.4 (Cardinality as a relation)**.** Let X be a finite set, i.e., a set which only contains finitely many elements. For $A \subseteq X$ let $\text{card}(A)$ be the number of its elements. [38] Let

$$R := \{(A, B) : A, B \subseteq X \text{ and } \text{card}(A) = \text{card}(B)\},$$

i.e., ARB if and only if A and B possess the same number of elements. This defines a relation on the power set 2^X of X which is reflexive, symmetric and transitive. $\square$

Example 4.5 (Empty relation)**.** Given two sets X and Y let $R := \emptyset$. This **empty relation** is the only relation on (X, Y) if X or Y is empty. $\square$

Example 4.6. Let $X := \mathbb{R}^2$ be the xy–plane. For any point $\vec{x} = (x_1, x_2)$ in the plane let $\|\vec{x}\| := \sqrt{x_1^2 + x_2^2}$ be its length [39] and let $R := \{(\vec{x}, \vec{y}) \in \mathbb{R}^2 \times \mathbb{R}^2 : \|\vec{x}\| = \|\vec{y}\|\}$. In other words, two points in the plane are related when they have the same length: they are located on a circle with radius $r = \|\vec{x}\| = \|\vec{y}\|$. The relation R is reflexive, symmetric and transitive but not antisymmetric. $\square$

The relations given in examples 4.2, 4.4, 4.5 and 4.6 are reflexive, symmetric and transitive. Such relations are so important that they deserve a special name:

Definition 4.3 (Equivalence relation and equivalence classes)**.** Let R be a relation on a set X which is reflexive, symmetric and transitive. We call such a relation an **equivalence relation** on X. It is customary to write $x \sim y$ rather than xRy (or $(x, y) \in R$) and we say that x and y are **equivalent**

Given an equivalence relation "$\sim$" on a set X and $x \in X$ let

$$[x]_\sim := \{y \in X : y \sim x\} = \{\text{ all items equivalent to } x\ \}. \tag{4.1}$$

We call $[x]_\sim$ the **equivalence class** of x. If it is clear from the context what equivalence relation is referred to then we simply write $[x]$ instead of $[x]_\sim$. $\square$

[38] You will see later that $\text{card}(X)$ is the cardinality of A (see def.**??** on p.**??**).

[39] See def.9.3 on p.154. of the length or Euclidean norm of a vector in n-dimensional space.

Relations which are reflexive, antisymmetric and transitive like the relation of example 4.3 (set inclusion) allow to compare items for "bigger" and "smaller" or "before" and "after". They also deserve a special name:

Definition 4.4 (Partial Order Relation)**.** Let R be a relation on a set X which is reflexive, antisymmetric and transitive. We call such a relation a **partial ordering** of X. or a **partial order relation** on X. [40] It is customary to write "$x \preceq y$" or "$y \succeq x$" rather than "xRy" for a partial ordering R. We say that "x before y" or "y after x".

If "$x \preceq y$" defines a partial ordering on X then $(X, \preceq)$ is called a **partially ordered set** set or a **POset**. □

Remark 4.3. The properties of a partial ordering can now be phrased as follows:

$$x \preceq x \text{ for all } x \in X \qquad \textbf{reflexivity}, \tag{4.2}$$

$$x \preceq y \text{ and } y \preceq x \quad \Rightarrow \quad y = x \qquad \textbf{antisymmetry}, \tag{4.3}$$

$$x \preceq y \text{ and } y \preceq z \quad \Rightarrow \quad x \preceq z \qquad \textbf{transitivity}. \ \square \tag{4.4}$$

Remark 4.4 (Partial orderings and reflexivity)**.** Note the following:

A. According to the above definition, the following are partial orderings of X:

1. $X = \mathbb{R}$ and $x \preceq y$ if and only if $x \leqq y$.
2. $X = 2^\Omega$ for some set Ω and $A \preceq B$ if and only if $A \subseteq B$ (example 4.3).
3. $X = \mathbb{R}$ and $x \succeq y$ if and only if $x \geqq y$.

B. The following relations are **not** partial orderings of X because none of them is reflexive.

4. $X = \mathbb{R}$ and $x \preceq y$ if and only if $x < y$.
5. $X = 2^\Omega$ for some set Ω and $A \preceq B$ if and only if $A \subset B$ (i.e., $A \subseteq B$ but $A \neq B$).
6. $X = \mathbb{R}$ and $x \succeq y$ if and only if $x > y$.

Note that each one of those three relations is antisymmetric. For example, let us look at $x < y$. It is indeed true that the premise $\big[x < y$ **and** $y < x\big]$ allows us to conclude that $y = x$ as there are no such numbers x and y and a premise that is known never to be true allows us to conclude anything we want!

C. An equivalence relation $\sim$ is a never a partial ordering of X except in the very uninteresting case where you have $x \sim y$ if and only if $x = y$.

D. A partial ordering of X, as any relation on X in general, is inherited by any subset $A \subseteq X$ as follows: Let $\preceq$ be a partial ordering on a set X and let $A \subseteq X$. We define a relation $\preceq_A$ on A as follows: Let $x, y \in A$. Then $x \preceq_A y$ if and only if $x \preceq y$. □

Definition 4.5 (Inverse Relation)**.** Let X and Y be two sets and $R \subseteq X \times Y$ a relation on (X, Y). Let

$$R^{-1} := \{ (y, x) : (x, y) \in R \}.$$

Clearly R^{-1} is a subset of $Y \times X$ and hence a relation on (Y, X). We call R^{-1} the **inverse relation** to the relation R. □

[40] Some authors, Dudley among them, do not include reflexivity into the definition of a partial ordering and then distinguish between **strict partial orders** and **reflexive partial orders**.

Example 4.7. Let $R := \{(x, x^3) : x \in \mathbb{R}\}$. Then R is the relation on $\mathbb{R}$ which represents the function $y = f(x) = x^3$. We obtain

$$R^{-1} = \{(x^3, x) : x \in \mathbb{R}\} = \{(y, y^{1/3}) : y \in \mathbb{R}\}.$$

In other words, R^{-1} represents the inverse function $x = f^{-1}(y) = y^{1/3}$. $\square$

4.2 Functions (Mappings) and Families

4.2.1 Some Preliminary Observations about Functions

Remark 4.5 (A layman's definition of a function)**.** We look at the set $\mathbb{R}$ of all real numbers [41] and the function $y = f(x) = \sqrt{4 - x^2}$ which associates with certain real numbers x (the "argument" or "independent variable") another real number $y = \sqrt{4 - x^2}$ (the "function value" or "dependent variable"):

$$f(0) = \sqrt{(4-0} = 2,\ f(2) = f(-2) = \sqrt{(4-4} = 0,\ f(2/3) = f(-2/3) = \sqrt{(36-4)/9} = \sqrt{30}/3,\ \dots$$

You can think of this function as a rule or law which specifies what item y is obtained as the output or result if the item x is provided as input.

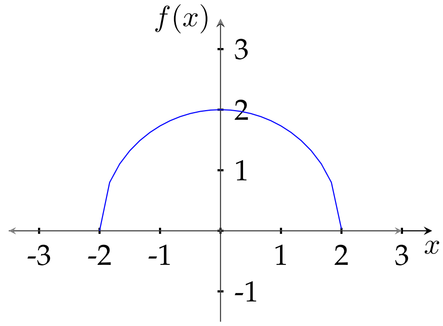
Let us look a little bit closer at the function $y = f(x) = \sqrt{4 - x^2}$ and its properties:

- **a**. For some real numbers x there is no function value: For example, if $x = 10$ then $4 - x^2 = -96$ is negative and the square root cannot be taken.
- **b**. For some other x, e.g., $x = 0$ or $x = 2/3$, there is a function value $f(x)$. A moment's reflection shows that the biggest possible set of potential arguments [42] is the interval $[-2, 2]$. It is customary to write D_f for the natural domain of a function $y = f(x)$.
- **c**. For a given x there is never more than one function value $f(x)$. This property allows us to think of a function as an assignment rule: It assigns to certain arguments x a unique function value $f(x)$. We observed in **b** that $f(x)$ exists if and only if $x \in [-2, 2]$.
- **d**. Not every $y \in \mathbb{R}$ is suitable as a function value: A square root cannot be negative, hence no x exists such that $f(x) = -1$ or $f(x) = -\pi$.

[41] Real numbers were defined in section "Numbers" on p.15.
[42] This set is called by some authors the **natural domain** of the function (e.g., [2] Brewster/Geoghegan).

e. On the other hand, there are numbers y such as $y = 2$, which are "hit" more than once by the function: $f(2) = f(-2) = 0$. [43]

f. Graphs as drawings: We are used to look at the graphs of functions, Here is a picture of the graph of $f(x) = \sqrt{4 - x^2}$.



g. Graphs as sets: Drawings as the one above have limited precision (the software should have drawn a perfect half circle with radius 2 about the origin but there seem to be wedges at $x \approx \pm 1.8$). Also, how would you draw a picture of a function which assigns a 3–dimensional vector [44] (x, y, z) to its distance $w = F(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ from the zero vector $(0, 0, 0)$? You would need four dimensions, one each for x, y, z, w, to draw the graph!

To express the graph of a function without a picture, let us look at a verbal description: The graph of a function $f(x)$ is the collection of the pairs $(x, f(x))$ for all points x which belong to the set $[-2, 2]$ of potential arguments (see **a**). In mathematical parlance: The graph of the function $f(x)$ is the set

$$\Gamma_f := \{(x, f(x)) : x \in D_f\}$$

(see remark 4.1 on p. 69).

□

We now make adjustments to some of those properties which will get us closer to the definition of a function as it is used in abstract mathematics.

Remark 4.6 (A better definition of a function)**.** We make the following alterations to remark 4.5.

- We require an upfront specification of the set A of items that will be allowed as input (arguments) for the function and we require that $y = f(x)$ makes sense for each $x \in A$. Given the function $y = f(x) = \sqrt{4 - x^2}$ from above this means that A must be a subset of $[-2, 2]$.
- We require an upfront specification of the set B of items that will be allowed as output (function values) for the function. This set must be so big that each $x \in A$ has a function value $y \in B$. We do not mind if B contains redundant y values. For $y = f(x) = \sqrt{4 - x^2}$ any superset of the closed interval $[0, 2]$ will do. We may choose $B := [0, 2]$ or $B := [-2, 2\pi]$ or $B := [0, 4]$ or $B := \mathbb{R} \cup \{$ all inhabitants of Chicago $\}$.

[43] Matter of fact, only for $y = 2$ there exists a single argument x such that $y = f(x)$ $(x = 0)$. All other y-values in the interval $[0, 2]$ are "mapped to" by two different arguments $x = \pm\sqrt{4 - y^2}$.

[44] Skip this example on first reading if you do not know about functions of several variables. You will find information about this in chapter 9 ("Vectors and vector spaces") on p.151.

Doing so gives us the following: A function consists of three items: a set A of inputs, a set B of outputs and an assignment rule $x \mapsto f(x)$ with the following properties:

1. For **all** inputs $x \in A$ there is a function value $f(x) \in B$.
2. For any input $x \in A$ there is never more than one function value $f(x) \in B$. It follows from property 1 that each $x \in A$ uniquely determines its function value $y = f(x)$. This property allows us to think of a function as an assignment rule: It assigns to each $x \in A$ a unique function value $f(x) \in B$.
3. Not every $y \in B$ needs to be a function value $f(x)$ for some $x \in A$, i.e., the set $\{x \in A : f(x) = y\}$ can be empty.
4. On the other hand there may be numbers y which are "hit" more than once by f.
   Example: Let $A := \mathbb{N}$, $B := \mathbb{R}$, $f(x) := (-1)^x$. Then both -1 and 1 are mapped to infinitely often by f. by f.
5. The graph Γ_f of a function $f(x)$ is the collection of the pairs $(x, f(x))$ for all points x which belong to the set A, i.e.,

$$\Gamma_f := \{(x, f(x)) : x \in A\}. \tag{4.5}$$

   Γ_f has the following properties:

5a. $\Gamma_f \subseteq A \times B$, i.e., Γ_f is a relation on (A, B) (see def.4.2 on p.70).
5b. For each $x \in A$ there exists a unique $y \in B$ such that $(x, y) \in \Gamma_f$
5c. If $x \mapsto g(x)$ is another function with inputs A and outputs B which is different from $x \mapsto f(x)$ (i.e., there is at least one $a \in A$ such that $f(a) \neq g(a)$) then the graphs Γ_f and Γ_g do not coincide

6. Conversely, if A and B are two nonempty sets, then any relation Γ on (A, B) which satifies **5a** and **5b** uniquely determines a function $x \mapsto f(x)$ with inputs A and outputs B as follows: For $a \in A$ we define $f(a)$ to be the element $b \in B$ for which $(a, b) \in \Gamma$. We know from **5b** that such b exists and is uniquely determined.

□

Here is a complicated way of looking at the example above: Let $X = [-2, 2]$ and $Y = \mathbb{R}$. Then $y = f(x) = \sqrt{4 - x^2}$ is a rule which "maps" each element $x \in X$ to a uniquely determined number $y \in Y$ which depends on x as follows: Subtract the square of x from 4, then take the square root of that difference.

Mathematicians are very lazy as far as writing is concerned and they figured out long ago that writing "depends on xyz" all the time not only takes too long, but also is aesthetically very unpleasing and makes statements and their proofs hard to understand. They decided to write "(xyz)" instead of "depends on xyz" and the modern notion of a function or mapping $y = f(x)$ was born.

Here is another example: if you say $f(x) = x^2 - \sqrt{2}$, it's just a short for "I have a rule which maps a number x to a value $f(x)$ which depends on x in the following way: compute $x^2 - \sqrt{2}$." It is crucial to understand from which set X you are allowed to pick the "arguments" x and it is often helpful to state what kinds of objects $f(x)$ the x–arguments are associated with, i.e., what set Y they will belong to.

We now are ready to give the precise definition of a function.

4.2.2 Definition of a Function and Some Basic Properties

We have seen in remark 4.6 on p. 73 that a function can be thought of equivalently as an assignment rule $x \mapsto f(x)$ or as a graph. Mathematicians prefer the latter because "assignment rule" is a rather vague term (an undefined term in the sense of ch. 3.7.1 (Building blocks of mathematical theories) on p.57) whereas "graph" is entirely defined in the language of sets. This chapter starts with the official definition of a function. It then deals with the following concepts: composition of functions, injective, surjective, bijective and inverse functions, restriction and extension of functions.

Definition 4.6 (Mappings (functions))**.** Given are two arbitrary nonempty sets X and Y and a relation Γ on (X, Y) (see 4.2 on p.70) which satisfies the following:

(4.6) for each $x \in X$ there exists exactly one $y \in Y$ such that $(x, y) \in \Gamma$.

We call the triplet $f(\cdot) := (X, Y, \Gamma)$ a **function** or **mapping** from X to Y. The set X is called the **domain** or **source** and Y is called the **codomain** or **target** of the mapping $f(\cdot)$. We will mostly use the words "domain" and "codomain" in this document.

> Usually mathematicians simply write f instead of $f(\cdot)$ We mostly follow that convention, but sometimes include the "$(\cdot)$" part to emphasize that a function rather than an "ordinary"' element of a set is involved.

Let $x \in X$. We write $f(x)$ for the uniquely determined $y \in Y$ such that $(x, y) \in \Gamma$. We write Γ_f or $\Gamma(f)$ if we want to stress that Γ is the relation associated with the function $f = (X, Y, \Gamma)$. We call Γ the **graph** of the function f. Clearly

$$\Gamma = \Gamma_f = \Gamma(f) = \{(x, f(x)) : x \in X\}. \tag{4.7}$$

It is customary to write

$$f : X \to Y, \qquad x \mapsto f(x) \tag{4.8}$$

instead of $f = (X, Y, \Gamma)$ and we henceforth follow that convention. We abbreviate that to $f : X \to Y$ if it is clear or irrelevant how to compute $f(x)$ from x. We read "$a \mapsto b$" as "a is assigned to b" or "a maps to b" and refer to $\mapsto$ as the **maps to operator** or **assignment operator**.

Domain elements $x \in X$ are called **independent variables** or **arguments** and $f(x) \in Y$ is called the **function value** of x. The subset

$$f(X) := \{y \in Y : y = f(x) \text{ for some } x \in X\} = \{f(x) : x \in X\} \tag{4.9}$$

of Y is called the **range** or **image** of the function $f(\cdot)$. [45]

We say "f maps X into Y" and "f maps the domain value x to the function value $f(x)$". □

Figure 4.1 on p.76 illustrates the graph of a function as a subset of $X \times Y$.

[45] We distinguish the target (codomain) Y of $f(\cdot)$ from its image (range) $f(X)$ which is a subset of Y.

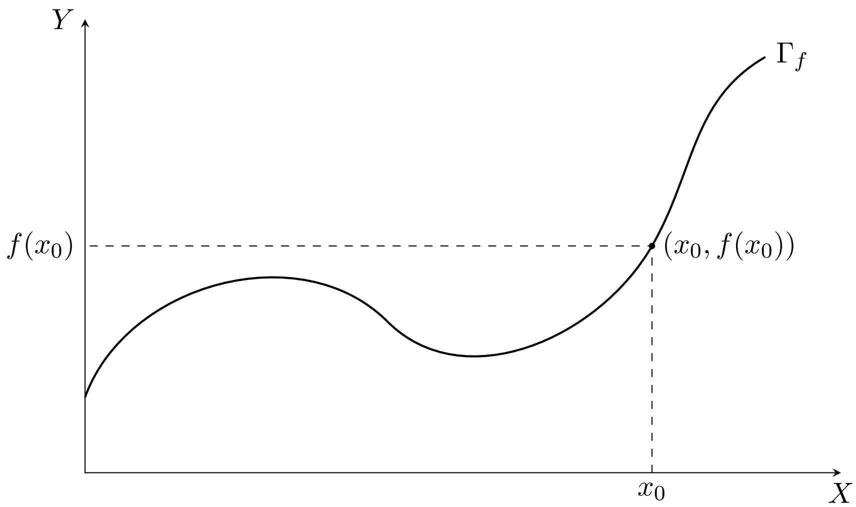


Figure 4.1: Graph of a function.

Remark 4.7 (Mappings vs. functions)**.** Mathematicians do not always agree 100% on their definitions. The issue of what is called a function and what is called a mapping is subject to debate. Some mathematicians call a mapping a function only if its codomain is a subset of the real numbers [46] but the majority does what I'll try to adhere to in this document: I use "mapping" and "function" interchangeably and I'll talk about **real functions** rather than just functions if the codomain is part of $\mathbb{R}$ (see (4.12) on p.87). $\square$

Remark 4.8. The symbol x chosen for the argument of the function is a **dummy variable** in the sense that it does not matter what symbol you use.

The following each define the same function with domain $[0, \infty[$ and codomain $\mathbb{R}$ which assigns to any non-negative real number its (positive) square root:

$$\begin{aligned} f : [0, \infty[\to \mathbb{R}, &\qquad x \mapsto \sqrt{x}, \\ f : [0, \infty[\to \mathbb{R}, &\qquad y \mapsto \sqrt{y}, \\ f : [0, \infty[\to \mathbb{R}, &\qquad f(\gamma) \ = \ \sqrt{\gamma}. \end{aligned}$$

Matter of fact, not even the symbol you choose for the function matters as long as the operation (here: assign a number to its square root) is unchanged. In other words, the following still describe the same function as above:

$$\begin{aligned} \varphi : [0, \infty[\to \mathbb{R}, &\qquad t \mapsto \sqrt{t}, \\ A : [0, \infty[\to \mathbb{R}, &\qquad x \mapsto \sqrt{x}, \\ g : [0, \infty[\to \mathbb{R}, &\qquad g(A) \ = \ \sqrt{A}. \end{aligned}$$

[46] or if the codomain is a subset of the complex numbers, but we won't discuss complex numbers in this document.

In contrast, the following three functions all are different from each other and none of them equals f because domain and/or codomain do not match:

$$\psi :]0, \infty[\to \mathbb{R}, \qquad x \mapsto \sqrt{x} \quad \text{(different domain)}.$$
$$B : [0, \infty[\to]0, \infty[, \qquad x \mapsto \sqrt{x} \quad \text{(different codomain)},$$
$$h : [0, 1[\to [0, 1[, \qquad x \mapsto \sqrt{x} \quad \text{(different domain and codomain)}. \quad \square$$

Definition 4.7 (Function composition)**.** Given are three nonempty sets X, Y and Z and two functions $f : X \to Y$ and $g : Y \to Z$. Given $x \in X$ we know the meaning of the expression $g\big(f(x)\big)$:

$y := f(x)$ is the function value of x for the function f, i.e., the unique $y \in Y$ such that $(x, y) \in \Gamma_f$.

$z := g(y) = g\big(f(x)\big)$ is the function value of $f(x)$ for the function g, i.e., the unique $z \in Z$ such that $\big(f(x), z\big) = \big(f(x), g(f(x))\big) \in \Gamma_g$.

The set $\Gamma := \{\big(x, g(f(x)) : x \in X\big)\}$ is a relation on (X, Z) such that

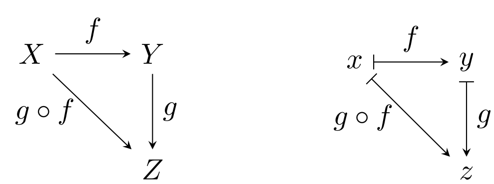
(4.10) for each $x \in X$ there exists exactly one $z \in Z$, namely, $z = g\big(f(x)\big)$, such that $(x, z) \in \Gamma$.

It follows that Γ is the graph of a function $h = (X, Z, \Gamma)$ with function values $h(x) = g\big(f(x)\big)$ for each $x \in X$. We call h the **composition** of f and g and we write $h = g \circ f$ ("g after f").

As far as notation is concerned it is OK to write either of $g \circ f(x)$ or $(g \circ f)(x)$. The additional parentheses may give a clearer presentation if f and/or g are fairly complex. $\square$

The following shows how you diagram the composition of two functions. The left picture shows the domains and codomains for each mapping and the left one the element assignments.

(4.11) *Function composition*



We have a special name for the "do nothing function" which assigns each argument to itself:

Definition 4.8 (identity mapping)**.** Given any non–empty set X, we use the symbol id_X for the **identity** mapping defined as

$$id_X : X \to X, \qquad x \mapsto x.$$

We drop the subscript if it is clear what set is referred to. $\square$

4.2.3 Examples of Functions

We now give some examples of functions. You might find some of them rather difficult to understand at first reading.

Example 4.8. Let $\Gamma := \{(x, x^3) : x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}$. Then $f = (\mathbb{R}, \mathbb{R}, \Gamma)$ is the function

$$f : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto x^3. \ \square$$

Example 4.9. Let $\Gamma := \{(x, x^2 + 1) : x \in \mathbb{R}\}$. Then $g = (\mathbb{R}, \mathbb{R}, \Gamma)$ is the function

$$g : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto x^2 + 1. \ \square$$

Example 4.10. Let $\Gamma := \{(a, \ln(a)) : a \in]0, \infty[\}$. Here $\ln(a)$ denotes the natural logarithm of a. Then $h = (]0, \infty[, \mathbb{R}, \Gamma)$ is the function

$$h :]0, \infty[\to \mathbb{R}, \qquad x \mapsto \ln(x). \ \square$$

Example 4.11. Let $\Gamma := \{(x, \sqrt{x}) : x \in [0, \infty[\}$. Then $\varphi = ([0, \infty[, \mathbb{R}, \Gamma)$ is the function

$$\varphi : [0, \infty[\to \mathbb{R}, \qquad x \mapsto \sqrt{x}. \ \square$$

Example 4.12. Let $\Gamma := \{(x, \sqrt{x}) : x \in [0, \infty[\}$. We can consider Γ as a subset of $[0, \infty[\times \mathbb{R}$ but also as a subset of $[0, \infty[\times [0, \infty[$. In the first case we obtain a function $\varphi = ([0, \infty[, \mathbb{R}, \Gamma)$, i.e., the function

$$\varphi : [0, \infty[\to \mathbb{R}, \qquad x \mapsto \sqrt{x}.$$

In the second case we obtain a different(!) function $\psi = ([0, \infty[, [0, \infty[, \Gamma)$, i.e., the function

$$\psi : [0, \infty[\to [0, \infty[, \qquad x \mapsto \sqrt{x}. \ \square$$

If you have taken multivariable calculus or linear algebra then you know that functions need not necessarily map numbers to numbers but they can also map vectors to numbers, numbers to vectors (curves) or vectors to vectors.

Example 4.13. We define a function which maps two-dimensional vectors to numbers. Let $A := \{((x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqq 1\}$. Let $\Gamma := \{((x, y), \sqrt{1 - x^2 - y^2}) : (x, y) \in A\}$. Then $F = (A, \mathbb{R}, \Gamma)$ is the function

$$F : A \to \mathbb{R}, \qquad (x, y) \mapsto \sqrt{1 - x^2 - y^2}.$$

Note that the domain is not a set of real numbers but of points in the plane and that the graph of F is a set of points (x, y, z) in 3–dimensional space. $\square$

Example 4.14. We define a function which maps numbers to two-dimensional vectors (a curve in the plane). Let $\Gamma := \{(t, (\sin t, \cos t)) : t \in \mathbb{R}\}$. Then $G = (\mathbb{R}, \mathbb{R}^2, \Gamma)$ is the function

$$G : \mathbb{R} \to \mathbb{R}^2, \qquad t \mapsto (\sin t, \cos t).$$

whose image $G(\mathbb{R}$ is the unit circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ Note that the codomain is not a set of real numbers but the Euclidean plane. $\square$

Example 4.15. Let $\Gamma := \{((x, y), (2x - y/3,\ x/6 + 4y)) : x, y \in \mathbb{R}\}$. Then $H = (\mathbb{R}^2, \mathbb{R}^2, \Gamma)$ is the function

$$H : \mathbb{R}^2 \to \mathbb{R}^2, \qquad (x, y) \mapsto (2x - y/3,\ x/6 + 4y).$$

Note that both domain and codomain are the Euclidean plane. $\square$

We now reformulate the last example in the framework of linear algebra. Skip this next example if you do not know about matrix multiplication.

Example 4.16. As is customary in linear algebra we now think of $\mathbb{R}^2$ as the collection of column vectors $\{\begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R}\}$ rather than the cartesian product $\mathbb{R}^2 \times \mathbb{R}^2$ which is the collection of row vectors $\{(x, y) : x, y \in \mathbb{R}\}$.

Let A be the 2×2 matrix

$$A := \begin{pmatrix} 2 & -1/3 \\ 1/6 & 4 \end{pmatrix}.$$

We then obtain for any pair of numbers $\vec{x} = (x, y)^T$ [47] that

$$A\vec{x} = \begin{pmatrix} 2 & -1/3 \\ 1/6 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - y/3 \\ x/6 + 4y \end{pmatrix}$$

Let $\Gamma := \{(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 2x - y/3 \\ x/6 + 4y \end{pmatrix}) : x, y \in \mathbb{R} \}$. Then $H = (\mathbb{R}^2, \mathbb{R}^2, \Gamma)$ is the function

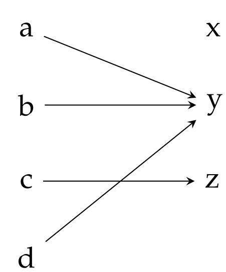
$$H : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \end{pmatrix}.$$

Note that both domain and codomain are the Euclidean plane. $\square$

If you want to construct a counterexample to a mathematical statement concerning functions it often is best to construct functions with small domain and codomain so that you can draw a picture that completely describes the assignments. The next example will illustrate this.

Example 4.17.

Let $X := \{a, b, c, d\}$, $Y := \{x, y, z\}$, $\Gamma := \{ (a, y), (b, y), (c, z), (d, y)\}$. Then $I = (X, Y, \Gamma)$ is the function which maps the elements of X to Y according to the diagram on the right. Note that nothing was said about the nature of the elements of X and Y. One need not know about it to make observations like the following: Examine items **3** and **4** of remark 4.6 (A better definition of a function) on p.73. Convince yourself that $x \in Y$ is an example for **3**: Not every element



of Y needs to be a function value and that $y \in Y$ is an example for **4**: There may be elements of Y which are "hit" more than once by the function. $\square$

Example 4.18. This example represents a mathematical model for computing probabilities of the outcomes of rolling a fair die and demonstrates that probability can be thought of as a function that maps sets to numbers.

[47] Here $(x, y)^T = \begin{pmatrix} x \\ y \end{pmatrix}$ is the **transpose** of $(x.y)$, i.e., the operation that switches rows and columns of any matrix. In particular it transforms a row vector into a column vector and vice versa.

If we roll a die then the outcome will be an integer between 1 and 6, i.e., the "state space" for this random action will be $X := \{1, 2, 3, 4, 5, 6\}$. For $A \subseteq X$ let $\text{Prob}(A)$ denote the probability that rolling the die results in an outcome $x \in A$.

For example Prob(an even number occurs) $= \text{Prob}\big(\{2, 4, 6\}\big) = 50\% = 1/2$. Clearly we have for singletons consisting of a single outcome that

$$\text{Prob}\big(\{1\}\big) = \text{Prob}\big(\{2\}\big) = \cdots = \text{Prob}\big(\{6\}\big) = 1/6 = 16.\bar{6}\%.$$

Your everyday experience tells you that if $A = \{x_1, x_2, \dots, x_k\}$ where $x_j \in X$ for each index j (and hence $k \leqq 6$ because a set does not contain duplicates) then

$$\text{Prob}(A) = \text{Prob}\big(\{x_1\}\big) + \text{Prob}\big(\{x_2\}\big) + \cdots + \text{Prob}\big(\{x_k\}\big) = \sum_{j=1}^{k} \text{Prob}\big(\{x_j\}\big).$$

What if A is the event that the roll of the die does not result in any outcome, i.e., $A = \emptyset$? We do not worry about the die getting stuck in mid-air or the dog snatching it before we get a chance to see the outcome and consider this event impossible, i.e., $\text{Prob}(\emptyset) = 0$.

We now have a probability associated with every $A \subseteq X$, i.e., with every $A \in 2^X$ and can finally write this probability as a function. Let $\Gamma := \{(A, \text{Prob}(A)) : A \subseteq X\}$. Then $P = (2^X, [0, 1], \Gamma)$ is the function

$$P : 2^X \to [0, 1], \qquad A \mapsto \text{Prob}(A).$$

Why do we use $[0, 1]$ and not $\mathbb{R}$ as the codomain? The answer is that we could have done so but no event has a probablity that exceeds 100% or is negative, so $[0, 1]$ is big enough and by choosing this set as the codomain we do not deviate from standard presentation of mathematical probability theory. $\square$

To understand the next example you need to be familiar with the concepts of continuity, differentiability and antiderivatives (integrals) of functions of a single variable. Just skip the parts where you lack the background.

Example 4.19. Let $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$ and let $X :=]a, b[$ be the open (end points a, b are excluded) interval of all real numbers between a and b. Let $x_0 \in]a, b[$ be "fixed but arbitrary".

The following is known from calculus (see [8] Stewart, J: Single Variable Calculus): Let $f :]a, b[\to \mathbb{R}$ be a function which is continuous on $]a, b[$. Then

a. f is integrable, i.e., for any $\alpha, \beta \in \mathbb{R}$ such that $a < \alpha < \beta < b$ the **definite integral** $\int_\alpha^\beta f(u)du$ exists. For a definition of integrability see section 4.1 of [8] Stewart, J: Single Variable Calculus.

b. Integration is "linear", i.e., it is additive: $\int_\alpha^\beta \big(f(u) + g(u)\big)du = \int_\alpha^\beta f(u)du + \int_\alpha^\beta g(u)du$, and you can "pull out" constant $\lambda \in \mathbb{R}$: $\int_\alpha^\beta \lambda f(u)du = \lambda \int_\alpha^\beta f(u)du$.

b. Integration is "monotone":
If $f(x) \leqq g(x)$ for all $\alpha \leqq x \leqq \beta$ then $\int_\alpha^\beta \big(f(u)\big)du \leqq \int_\alpha^\beta g(u)du$.

b. f has an **antiderivative** : There exists a function $F :]a,b[\to \mathbb{R}$ whose derivative $F'(\cdot)$ exists on all of $]a,b[$ and coincides with f, i.e., $F'(x) = f(x)$ for all $x \in]a,b[$.

c. This antiderivative satisfies $\displaystyle\int_\alpha^\beta f(u)du \;=\; F(\beta) - F(\alpha)$ for all $a < \alpha < \beta < b$ and it is **not** uniquely defined: If $C \in \mathbb{R}$ then $F(\cdot) + C$ is also an antiderivative of f.
On the other hand, if both F_1 and F_2 are antiderivatives for f then their difference $G(\cdot) := F_2(\cdot) - F_1(\cdot)$ has the derivative $G'(\cdot) = f(\cdot) - f(\cdot)$ which is constant zero on $]a,b[$. It follows that any two antiderivatives only differ by a constant.
To summarize the above: If we have one antiderivative F of f then any other antiderivative $\tilde{F}$ is of the form $\tilde{F}(\cdot) = F(\cdot) + C$ for some real number C.
This fact is commonly expressed by writing $\displaystyle\int f(u)du \;=\; F(x) + C$ for the **indefinite integral** (an integral without integration bounds).

d. It follows from **c** that if some $c_0 \in \mathbb{R}$ is given then there is only one antiderivative F such that $F(x_0) = c_0$.
Here is a quick proof: Let G be another antiderivative of f such that $G(x_0) = c_0$. Because $G - F$ is constant we have for all $x \in]a,b[$ that

$$G(x) - F(x) \;=\; \text{const} \;=\; G(x_0) - F(x_0) \;=\; 0,$$

i.e., $G = F$.

After those reminders about integration we are ready to define a function $I(\cdot)$ for which both domain and codomain are sets of functions.

Given are a, b and x_0 as above and $c_0 \in \mathbb{R}$. Let

$$\begin{aligned}\mathscr{F} &:= \{ f :]a,b[\to \mathbb{R} \text{ such that } f \text{ is continuous on }]a,b[\ \},\\ \mathscr{G} &:= \{ g :]a,b[\to \mathbb{R} \text{ such that } g \text{ is differentiable on }]a,b[\text{ and } g(x_0) = c_0\ \}.\end{aligned}$$

It follows from the preparatory remarks that for each $f \in \mathscr{F}$ there exists a unique $F \in \mathscr{G}$ which is an antiderivative for f. We now define a function $I : \mathscr{F} \to \mathscr{G}$ by specifying its graph as the set

$$\Gamma \;:=\; \{ \big(f(\cdot), g(\cdot)\big) : f \in \mathscr{F}, g \in \mathscr{G} \text{ and } g \text{ is an antiderivative of } f\ \}.$$

In other words, $I = (\mathscr{F}, \mathscr{G}, \Gamma)$ is the function

$$I : \mathscr{F} \to \mathscr{G}, \qquad f(\cdot) \mapsto I(f)(\cdot) = \int f(u)du + C$$

where C is determined by the requirement that $I(f)(x_0) = c_0$. □

4.2.4 Injective, Surjective and Bijective functions

Definition 4.9 (Surjective, injective, bijective)**.** Let $f : X \to Y$. As usual the graph of f is denoted Γ_f.

a. Surjectivity: In general it is not true that $f(X) = \{f(x) : x \in X\}$ equals the entire codomain Y, i.e., that

(4.12) for each $y \in Y$ there exists at least one $x \in X$ such that $(x,y) \in \Gamma_f$.

But if $f(X) = Y$, i.e., if (4.12) holds, we call f **surjective** and we say that f maps X **onto** Y.

b. Injectivity: In general it is not true that if $y \in f(X)$ then $y = f(x)$ for a unique x, i.e., that if there is another $x_1 \in X$ such that also $y = f(x_1)$ then it follows that $x_1 = x$. But if this is the case, i.e., if

(4.13) for each $y \in Y$ there exists at most one $x \in X$such that $(x, y) \in \Gamma_f$.

then we call f **injective** .

We can express (4.13) also as follows: If $x, x_1 \in X$ and $y \in Y$ are such that $(x, y) \in \Gamma_f$ and $(x_1, y) \in \Gamma_f$ then it follows that $x_1 = x$.

c. Bijectivity: Let $f : X \to Y$ be both injective and surjective. Such a function is called **bijective**.

It follows from (4.12) and (4.13) that f is bijective if and only if

(4.14) for each $y \in Y$ there exists exactly one $x \in X$such that $(x, y) \in \Gamma_f$.

We rewrite (4.14) by employing Γ_f's inverse relation $\Gamma_f^{-1} = \{(y, x) : (x.y) \in \Gamma\}$ (see def. 4.5 on p.71) and obtain

(4.15) for each $y \in Y$ there exists exactly one $x \in X$such that $(y, x) \in \Gamma_f^{-1}$.

But this implies, according to (4.6), that Γ_f^{-1} is the graph of a function $g := (Y, X, \Gamma_f^{-1})$ with domain Y and codomain X where, for a given $y \in Y$, $g(y)$ stands for the uniquely determined $x \in X$ such that $(y, x) \in \Gamma_f^{-1}$. Note that

$$\Gamma_f^{-1} = \Gamma_g. \tag{4.16}$$

We call g the **inverse mapping** or **inverse function** of f and write f^{-1} instead of g. □

Remark 4.9.

a. It follows from (4.16) that

$$\Gamma_f^{-1} = \Gamma_{f^{-1}}. \tag{4.17}$$

b. Each $x \in X$ is mapped to $y = f(x)$ which is the only element of Y such that $f^{-1}(y) = x$,

c. Each $y \in Y$ is mapped to $x = f^{-1}(y)$ which is the only element of X such that $f(x) = y$.

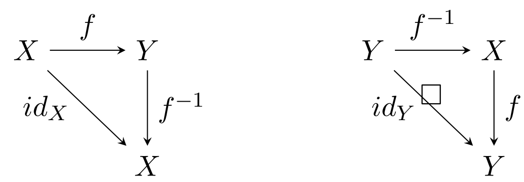
d. It follows from **b** and **c** that

$$\text{if } x \in X, y \in Y \text{ then } f(x) = y \iff y = f^{-1}(y). \tag{4.18}$$

e. It also follows from **b** and **c** that $f^{-1}(f(x)) = x$ for all $x \in X$ and $f(f^{-1}(y)) = y$ for all $y \in Y$.

In other words, $f^{-1} \circ f = id_X$ and $f \circ f^{-1} = id_Y$. Here is the picture:

(4.19) Inverse function



Theorem 4.1 (Characterization of inverse functions)**.** *Let X and Y be nonempty sets and $f : X \to Y$. Then the following are equivalent:*

a. *f is bijective.*
b. *There exists $g : Y \to X$ such that both $g \circ f = id_X$ and $f \circ g = id_Y$.*

*Proof of **a** $\Rightarrow$ **b**: We have seen in part **d** of remark 4.9 that $g := f^{-1}$ satisfies **b**.*

*Proof of **b** $\Rightarrow$ **a**: We must show that f is both surjective and injective. First we show that f is surjective. Let $y \in Y$. we must find some $x \in X$ such that $f(x) = y$. Let $x := g(y)$. Then*

$$f(x) \;=\; f\big(g(y)\big) \;=\; f \circ g(y) \;=\; id_Y(y) \;=\; y.$$

We have $f(x) = y$ and this proves surjectivity. Now we show that f is injective. Let $x_1, x_2 \in X$ and $y \in Y$ such that $f(x_1) = f(x_2) = y$. We are done if we can prove that $x_1 = x_2$. We have

$$x_1 = id_X(x_1) = g \circ f(x_1) = g\big(f(x_1)\big) = g(y) = g\big(f(x_2)\big) = g \circ f(x_2) = id_X(x_2) = x_2,$$

i.e., $x_1 = x_2$. This proves injectivity of f. ■

Remark 4.10. [Horizontal and vertical line tests] Let X and Y be nonempty sets and $f : X \to Y$. The following needs to be taken with a grain of salt because X and Y may not be sets of real numbers.

Let $R \subseteq X \times Y$.

a. (4.6) on p.75 states that R is the graph of a function with domain X and codomain Y if and only if it passes the "vertical line test": Any "vertical line", i.e., any subset of $X \times Y$ of the form $V(x_0) := \{(x_0, y) : y \in Y\}$ for a fixed $x_0 \in X$ intersects R in **exactly one** point.
b. (4.12) on p.81 states that R is the graph of a surjective function with domain X and codomain Y if and only if it passes in addition to the vertical line test the following "horizontal line test": any "horizontal line", i.e., any subset of $X \times Y$ of the form $H(y_0) := \{(x, y_0) : x \in X\}$ for a fixed $y_0 \in Y$ intersects R in **at least one** point.
c. (4.13) on p.82 states that R is the graph of an injective function with domain X and codomain Y if and only if it passes in addition to the vertical line test the following horizontal line test: any "horizontal line", i.e., any subset of $X \times Y$ of the form $H(y_0) := \{(x, y_0) : x \in X\}$ for a fixed $y_0 \in Y$ intersects R in **at most one** point.
d. It follows from (4.14) on p.82 but also from the above that that R is the graph of a bijective function with domain X and codomain Y if and only if it passes in addition to the vertical line test the following horizontal line test: any "horizontal line", i.e., any subset of $X \times Y$ of the form $H(y_0) := \{(x, y_0) : x \in X\}$ for a fixed $y_0 \in Y$ intersects R in **exactly one** "point". Note the symmetry between this test and the one for vertical lines. The above is another indication that the inverse graph R^{-1} of a bijective function is a graph of a function (the inverse function f^{-1}). □

Remark 4.11. Abstract math is about proving theorems and propositions and functions are very important tools for that. It may be very important to know or to show that a certain function is injective or surjective or both. But these properties depend on the choice of domain and codomain as this simple example shows.

Let $f : A \to B$ be the function $f(x) := x^2$.

$A = \mathbb{R}, B = \mathbb{R}$:	f is neither injective nor surjective
$A =]-2, 3[, B = [0, 9[$:	f is surjective but not injective
$A =]0, 3[, B = [0, 9]$:	f is injective but not surjective
$A =]0, 3[, B =]0, 9[$:	f is bijective

The above demonstrates why domain and codomain are important for the specification of a function. $\square$

Proposition 4.1. *Let $X, Y, Z \neq \emptyset$. Let $f : X \to Y$ and $g : Y \to Z$.*

- ***a.*** *If both f, g are injective then $g \circ f$ is injective.*
- ***b.*** *If both f, g are surjective then $g \circ f$ is surjective.*
- ***c.*** *If both f, g are bijective then $g \circ f$ is bijective.*

*The proof of **a** and **b** is left as exercise 4.6.*

*Proof of **c**: Follows from **a** and **b** because bijective = injective + surjective.* $\blacksquare$

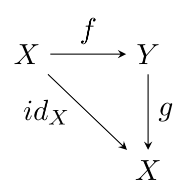
Corollary 4.1. *Let $X, Y, Z \neq \emptyset$. Let $f : X \to Y$ and $g : Y \to Z$.*

- ***a.*** *If f is bijective and g is injective then both $g \circ f$ and $f \circ g$ are injective.*
- ***b.*** *If f is bijective and g is surjective then both $g \circ f$ and $f \circ g$ are surjective.*
- ***c.*** *If f is bijective and g is bijective then both $g \circ f$ and $f \circ g$ are bijective.*

Proof:
***a** follows from prop.4.1.**a** because bijective functions are injective. **b** follows from prop.4.1.**b** because bijective functions are surjective. **c** follows from prop.4.1.**c**.* $\blacksquare$

We now examine conditions under which there are functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = id_X$, i.e.,

(4.20) $$g\big(f(x)\big) = x \text{ for all } x \in X :$$



Proposition 4.2. *Let $X, Y \neq \emptyset$. Let $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = id_X$. Then*

- ***a.*** *f is injective,*
- ***b.*** *g is surjective.*

*Proof of **a**: Let $x_1, x_2 \in X$. If $f(x_1) = f(x_2)$ then*

$$x_1 = id_X(x_1) = g(f(x_1)) = g(f(x_2)) = id_X(x_2) = x_2.$$

This proves injectivity of f.

*Proof of **b**: Let $x_0 \in X$. Let $y := f(x_0)$. Then $g(y) = g(f(x_0)) = g \circ f(x_0) = x_0$. We found for an arbitrary x_0 in the codomain of g some y which maps to x_0 and we have proved surjectivity of g.* ■

Proposition 4.3. *Let $X, Y \neq \emptyset$.*

- ***a.*** *Let $f : X \to Y$. If f is injective then there exists $g : Y \to X$ such that $g \circ f = id_X$ and any such function g is necessarily surjective.*
- ***b.*** *Let $g : Y \to X$. If g is surjective then there exists $f : X \to Y$ such that $g \circ f = id_X$ and any such function f is necessarily injective.*

*Proof of **a**: Let $Y' := f(X)$ and*

$$f' : X \to Y', \qquad x \mapsto f(x),$$

i.e., $f(x) = f'(x)$ for all $x \in X$. The only difference between f and f' is that we shrunk the codomain from Y to $f(X)$, thus making f' not only injective but also surjective, hence bijective. It follows that the inverse $(f')^{-1} : Y' \to X$ exists.

Let x_0 be an arbitrary, but fixed, element of X. We define $g : Y \to X$ as follows.

$$g(y) \ := \begin{cases} (f')^{-1}(y) & \text{if } y \in Y', \\ x_0 & \text{if } y \notin Y'. \end{cases}$$

*Let $x \in X$. Then $f(x) \in Y'$, hence $g \circ f(x) = g \circ f'(x) = (f')^{-1}\big(f'(x)\big) = x$. This proves $g \circ f = id_X$. We observe that g is surjective according to prop.4.2**a**.*

*Proof of **b**: For $x \in X$ let $Y_x := \{y \in Y : g(y) = x\}$. It follows from the surjectivity of g that $Y_x \neq \emptyset$ and we may select for each $x \in X$ some $y_x \in Y_x$.* [48]

Let $f : X \to Y$ be the function $x \mapsto y_x$. Let $x \in X$. Then

$$g \circ f(x) \ = \ g(y_x) \ = \ x.$$

*The first equality follows from the definition of f and the second one is true because $y_x \in Y_x$. It follows from prop.4.2**b** that f is injective.* ■

There are special names for functions f and g which are related by (4.20).

Definition 4.10 (Left inverses and right inverses)**.** Let $X, Y \neq \emptyset$. Let $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = id_X$. We call g a **left inverse** of f and we call f a **right inverse** of g. □

We combine that last definitions with the preceding two proposition and obtain

Theorem 4.2. *Let $X, Y \neq \emptyset$.*

- ***a.*** *Let $f : X \to Y$. Then f is injective $\Leftrightarrow$ f has a left inverse which is necessarily surjective.*
- ***b.*** *Let $g : Y \to X$. Then g is surjective $\Leftrightarrow$ g has a right inverse which is necessarily injective.*
- ***c.*** *An injection $X \to Y$ exists $\Leftrightarrow$ a surjection $Y \to X$ exists.*

[48] The ability to pick do such a selection regardless of the nature of X, Y and the surjective function $f : X \to Y$ is not something one can prove. It requires acceptance of the **Axiom of Choice**. See remark **??** on p.**??** in ch.**??** (Applications of Zorn's Lemma).

*Proof of **a** $\Rightarrow$): prop.4.3**a**.*
*Proof of **a** $\Leftarrow$): prop.4.2**a**.*

*Proof of **b** $\Rightarrow$): prop.4.3**b**.*
*Proof of **a** $\Leftarrow$): prop.4.2**b**.*

*Proof of **c** $\Rightarrow$): Let $f : X \to Y$ be injective. According to part **a** there exists a left inverse $g : Y \to X$ which is surjective*
*Proof of **c** $\Leftarrow$): Let $g : Y \to X$ be surjective. According to part **b** there exists a right inverse $f : X \to Y$ which is injective* ■

Remark 4.12. Let X and Y be two non-empty sets. No assumptions are made concerning how X and Y might be related.

a. Let $a \in Y$. Then the function

$$f : X \xrightarrow{\sim} \{a\} \times X; \quad x \mapsto (a, x) \tag{4.21}$$

is bijective because f has the function $(a, x) \mapsto x$ as an inverse.

b. The sets $\{0\} \times X$ and $\{1\} \times Y$ are disjoint. □

c. An injection/surjection/bijection $X \to Y$ exists if and only if an injection/surjection/bijection $\{0\} \times X \to \{1\} \times Y$ exists. □

Definition 4.11 (Restriction/Extension of a function)**.** Given are three non-empty sets A, X and Y such that $A \subseteq X$. Let $f : X \to Y$ a function with domain X. We define the **restriction of f to A** as the function

$$f\big|_A : A \to Y \quad \text{defined as} \quad f\big|_A(x) := f(x) \text{ for all } x \in A. \tag{4.22}$$

Conversely let $f : A \to Y$ and $\varphi : X \to Y$ be functions such that $f = \varphi\big|_A$. We then call φ an **extension** of f to X. □

Example 4.20. For an example let $X := \mathbb{R}$, $A := [0, 1]$ and $f(x) := 3x^2 (x \in [0, 1])$. For any $\alpha \in \mathbb{R}$ the function $\varphi_\alpha : \mathbb{R} \to \mathbb{R}$ defined as $\varphi_\alpha(x) := 3x^2$ if $0 \leqq x \leqq 1$ and αx otherwise defines a different extension of f to $\mathbb{R}$. □

Notations 4.1. As the only difference between f and $f\big|_A$ is the domain, it is customary to write f instead of $f\big|_A$ to make formulas look simpler if doing so does not give rise to confusions. □

Remark 4.13. The restriction $f\big|_A$ is always uniquely determined by f. Such is not the case for extensions if A is a strict subset of X unless some conditions are imposed on the nature of the extension.

For example, if we had asked for the continuity of the extension φ_α of f in example 4.20 above, only $\varphi_1(x) = 3x^2$ if $0 \leqq x \leqq 1$ and x otherwise would qualify. □

Many more properties of mappings will be discussed later. Now we look at families, sequences and some additional properties of sets.

4.2.5 Operations on Real Functions

Definition 4.12 (real functions)**.** Let X be an arbitrary, nonempty set. If the codomain Y of a mapping

$$f : X \to Y; \qquad x \mapsto f(x)$$

is a subset of $\mathbb{R}$, then we call $f(\cdot)$ a **real function** or **real valued function**. $\square$

Remember that this definition does not exclude the case $Y = \mathbb{R}$ because $Y \subseteq \mathbb{R}$ is in particular true if both sets are equal.

Real functions are a pleasure to work with because, given any fixed argument x_0, the object $f(x_0)$ is just an ordinary number. In particular you can add, subtract, multiply and divide real functions. Of course, division by zero is not allowed:

Definition 4.13 (Operations on real functions)**.** Let X be an arbitrary non-empty set.
Given are two real functions $f(\cdot), g(\cdot) : X \to (R)$ and a real number α. The **sum** $f + g$, **difference** $f - g$, **product** fg or $f \cdot g$, **quotient** f/g, and **scalar product** αf are defined by doing the operation in question with the numbers $f(x)$ and $g(x)$ for each $x \in X$. In other words these items are defined by the following equations:

$$\begin{aligned}
(f+g)(x) &:= f(x) + g(x),\\
(f-g)(x) &:= f(x) - g(x),\\
(fg)(x) &:= f(x)g(x),\\
(f/g)(x) &:= f(x)/g(x) \qquad \text{for all } x \in X \text{ where } g(x) \neq 0,\\
(\alpha f)(x) &:= \alpha \cdot g(x).
\end{aligned} \tag{4.23}$$

$\square$

Before we list some basic properties of addition and scalar multiplication of functions (the operations that interest us the most), let us have a quick look at constant functions.

Definition 4.14 (Constant functions)**.** Let a be an ordinary real number. You can think of a as a function from any non-empty set X to $\mathbb{R}$ as follows:

$$a(\cdot) : X \to \mathbb{R}; \qquad x \mapsto a.$$

In other words, the function $a(\cdot)$ assigns to each $x \in X$ one and the same value a. We call such a function which only takes a single value a **constant function**.

The most important constant real–valued function is the **zero function** $0(\cdot)$ which maps any $x \in X$ to the number zero. We usually just write 0 for this function unless doing so would confuse the reader. Note that scalar multiplication $(\alpha f)(x) = \alpha \cdot g(x)$ is a special case of multiplying two functions $(gf)(x) = g(x)f(x)$: Let $g(x) = \alpha$ for all $x \in X$ (constant function α).

The concept of a constant function makes sense for an arbitrary, nonempty codomain Y (i.e., Y need not be a set of real numbers):

We call any mapping f from X to Y a **constant function**. if its image $f(X) \subseteq Y$ is a singleton, i.e, it consists of exactly one element. $\square$

One last definition before we finally get so see some examples:

Definition 4.15 (Negative function)**.** Let X be an arbitrary, non-empty set and let $f : X \to \mathbb{R}$. The function

$$-f(\cdot) : X \to \mathbb{R}; \qquad x \mapsto -f(x).$$

is called **negative** f or **minus** f. We usually write $-f$ for $-f(\cdot)$. $\square$

All those last definitions about sums, products, scalar products, ... of real functions are very easy to understand if you remember that, for any fixed $x \in X$, you just deal with ordinary numbers!

Example 4.21 (Arithmetic operations on real functions)**.**
For simplicity, let $X := \mathbb{R}_+ = \{x \in \mathbb{R} : x \geqq 0\}$. Let

$$\begin{aligned}
f &: \mathbb{R}_{\geqq 0} \to \mathbb{R}; &\quad x &\mapsto (x-1)(x+1)\\
g &: \mathbb{R}_{\geqq 0} \to \mathbb{R}; &\quad x &\mapsto (x-1)\\
h &: \mathbb{R}_{\geqq 0} \to \mathbb{R}; &\quad x &\mapsto (x+1)
\end{aligned}$$

Then

$$\begin{aligned}
(f+h)(x) &= (x-1)(x+1) + x + 1 \ = x^2 - 1 + x + 1 \ = \ x(x+1)\ \forall x \in \mathbb{R}_+,\\
(f-g)(x) &= (x-1)(x+1) - (x-1) \ = x^2 - 1 - x + 1 \ = \ x(x-1)\ \forall x \in \mathbb{R}_+,\\
(gh)(x) &= (x-1)(x+1) \ = f(x) \ \forall x \in \mathbb{R}_+,\\
(f/h)(x) &= (x-1)(x+1)/(x+1) \ = x - 1 \ = \ g(x)\ \forall x \in \mathbb{R}_+,\\
(f/g)(x) &= (x-1)(x+1)/(x-1) \ = x + 1 \ = \ h(x)\ \forall x \in \mathbb{R}_+. \setminus \{1\}
\end{aligned}$$

It is really, really important for you to understand that f/g and h are **not the same functions** on $\mathbb{R}_+$. Matter of fact, f/g is not defined for all $x \in \mathbb{R}_+$ because for $x = 1$ you obtain $\frac{(1-1)(1+1)}{1-1} = 0/0$. The domain of f/g is different from that of h and both functions thus are different. $\square$

4.2.6 Sequences, Families and Functions as Families

Definition 4.16 (Indices)**.** Given is an expression of the form

$$X_\imath.$$

We say that X is **indexed by** or **subscripted by** or **tagged by** $\imath$. We call $\imath$ the **index** or **subsript** of X and we call $X_\imath$ an **indexed item** . $\square$

Remark 4.14. Both X and $\imath$ can occur in many different ways. Here is a collection of indexed items:

$$x_7,\ A_\alpha,\ k_T,\ \mathfrak{H}_{2/9},\ f_x,\ x_t, h_{\mathscr{A}},\ , i_{\mathbb{R}},\ H_{2\pi}$$

Some of the indices in this collection are highly unusual: Not only are some of them negative but they are fractions (e.g., $2/9$) or irrational (e.g., 2π) Others don't even look like numbers (e.g., α, T, x, t, $\mathscr{A}$ and $\mathbb{R}$). It is not clear from the information available to us whether those indices are names of variables which represent numbers or whether they represent functions, sets or other mathematical objects. There is one exception: The index $\mathbb{R}$ of $i_{\mathbb{R}}$ denotes the set of all real numbers. $\square$

We can turn any set into a "family" by tagging each of its members with an index. As an example, look at the following two indexed versions of the set S_2 from example 2.1 on p. 10:

$$\begin{aligned} F &= (a_1, e_1, e_2, i_1, i_2, i_3, o_1, o_2, o_3, o_4, u_3, u_5, u_9, u_{11}, u_{99}) \\ G &= (a_k, e_{-\sqrt{2}}, e_1, i_{-6}, i_{\mathscr{B}}, i_{\mathbb{R}}, o_7, o_{2/3}, o_{-8}, o_3, u_A, u_B, u_C, u_D, u_E) \end{aligned}$$

We note several things:

a. *F has the kind of indices that we are familiar with: all of them are positive integers.*
b. *Some of the indices in F occur multiple times. For example, 3 occurs as an index for i_3, o_3, u_3.*
c. *All of the indices in G are unique.*
d. *As in remark 4.14, some of the indices are very unusual.*

*The last point is not much of a problem as mathematicians are used to very unusual notation but point (**b**), the non-uniqueness of indices, is something that we want to avoid. From now on we ask for the following: The indices of an indexed collection must belong to some set J and each index $\imath \in J$ must be used exactly once. Remember that this automatically takes care of the duplicate indices problem as a set never contains duplicate values (see def.2.1 on p. 10). We also demand that there is a set X such that each indexed item $x_\imath$ belongs to X.*

We now are ready to give the definition of a family:

Definition 4.17 (Indexed families)**.** Let J and X be non-empty set and assume that

for each $\imath \in J$ there exists exactly one indexed item $x_\imath \in X$.

Let $R := \{(\imath, x_\imath) : \imath \in J\}$. Then R is a relation on (J, X) which satisfies (4.6) of the definition of a function

$$F : J \to X, \qquad \imath \mapsto F(\imath) := x_\imath$$

(see def.4.6 on p.75) whose graph Γ_F equals R.

We write $(x_\imath)_{\imath \in J}$ for this function if we want to deal with the collection of indexed elements $x_\imath$ rather than the function $F(\cdot)$ or the relation R. Reasons for this will be given in rem.4.16 on p.90.

$(x_\imath)_{\imath \in J}$ is called an **indexed family** or simply a **family** in X and J is called the **index set** of the family. For each $\jmath \in J$, $x_\jmath$ is called a **member of the family** $(x_\imath)_{\imath \in J}$.

$\imath$ is a dummy variable: $(x_\imath)_{\imath \in J}$ and $(x_k)_{k \in J}$ describe the same family as long as $\imath \mapsto x_\imath$ and $k \mapsto x_k$ describe the same function $F : J \to X$. This should not come as a surprise to you if you recall remark 4.8 on p.76. □

Remark 4.15. The codomain X does not occur in the notation $(x_\imath)_{\imath \in J}$. This is not a problem because we do not care about surjectivity or injectivity of families. The only thing that matters about the set X is that it is big enough to contain each indexed item. Here are two possible choices for a codomain.

a. If there is a universal set X which contains all tagged items of the family then X is acceptable as the codomain.

b. If there is no universal set then you can think of

$$X = \bigcup [x_\imath : \imath \in J] := \{x : x = x_{\imath_0} \text{ for some } \imath_0 \in I\}$$

as the codomain. [49] □

Definition 4.18 (Equality of families)**.** Two families $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$ are equal if

a. $I = J$,

b. $x_i = y_i$ for all $i \in I$. □

Note 4.1 (Simplified notation for families)**.** If there is no confusion about the index set then it can be dropped from the specification of a family and we simply write $(x_\imath)_\imath$ instead of $(x_\imath)_{\imath \in J}$. Even the index outside the right parentheses may be omitted and we write $(x_\imath)$ if it is clear that we are talking about families.

For example, a proposition may start as follows: Let (A_α) and (B_α) be two families of subsets of Ω indexed by the same set. Then .....

It is clear from the formulation that we deal in fact with two families $(A_\alpha)_{\alpha \in J}$ and $(B_\alpha)_{\alpha \in J}$. Nothing is said about J, probably because the proposition is valid for any index set or because this set was fixed once and for all earlier on for the entire section. □

Example 4.22. Here is an example of a family of subsets of $\mathbb{R}$ which are indexed by real numbers: Let $J = [0,1]$ and $X := 2^{\mathbb{R}}$. For $0 \leqq x \leqq 1$ let $A_x := [x, 2x]$ be the set of all real numbers between x and $2x$. Then $(A_x)_{x \in [0,1]}$ is such a family. □

Remark 4.16. If a family is just some kind of function, why bother with yet another definition? The answer to this is that there are many occasions in which writing something as a collection of indexed items rather than as a function makes things easier to understand. For example, take a peek at theorem 5.1 (De Morgan's Law) on p.98. One of the formulas there states that for any indexed family $(A_\alpha)_{\alpha \in I}$ of subsets of a universal set Ω it is true that

$$\Big(\bigcup_\alpha A_\alpha\Big)^{\complement} = \bigcap_\alpha A_\alpha^{\complement}.$$

Without the notion of a family you might have to say something like this: Let $A : I \to 2^\Omega$ be a function which assigns its arguments to subsets of Ω . Then

$$\Big(\bigcup_\alpha A(\alpha)\Big)^{\complement} = \bigcap_\alpha A(\alpha)^{\complement}.$$

The additional parentheses around the index α just add complexity to the formula. □

Example 4.23 (Sequences as families)**.** You have worked with special families before: those where $J = \mathbb{N}$ or $J = \mathbb{Z}_{\geq 0}$ and X is a subset of the real numbers. Example: $x_n := 1/n$. Here

$$(x_n)_{n \in \mathbb{N}} \text{ corresponds to the indexed collection } 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots \quad \square$$

[49] General unions and intersections will be defined in ch.5.1 (More on set operations). See def.5.1 on p.95.

Such families are called sequences:

Definition 4.19 (Sequences and subsequences)**.** Let $n_\star$ be an integer and let $J := \{n_\star, n_\star + 1, n_\star + 2, \dots\} = \{k \in \mathbb{Z} : k \geq n_\star\}$. Let X be an arbitrary nonempty set. An indexed family in X with index set J is called a **sequence** in X with **start index** $n_\star$.

Let I be an infinite subset of J. Because any two different integers m and n can be compared, i.e., either $m < n$ or $n < m$, [50] there are integers

$$n_1 < n_2 < n_3 < \dots \tag{4.24}$$

such that $I = \{n_j : j \in \mathbb{N}\}$. An indexed family in X with index set I is called a **subsequence** of the original sequence.

As for families, the name of the index variable of a sequence does not matter as long as it is applied consistently. It does not matter whether you write $(x_j)_{j\in J}$ or $(x_n)_{n\in J}$ or $(x_\beta)_{\beta\in J}$. □

Note 4.2 (Simplified notation for sequences)**.**

- **a.** It is customary to choose either of i, j, k, l, m, n as the symbol of the index variable of a sequence and to stay away from other symbols whenever possible.
- **b.** If J is defined as above then is not unusual to see "$(x_n)_{n\geq n_\star}$" instead of "$(x_n)_{n\in J}$". By default the index set for a sequence is $J = \mathbb{N} = \{1, 2, 3, 4, \dots\}$ and we can then write $(x_n)_n$ or just (x_n)
- **c.** Customary notation for subsequences is either of $(x_{n_j})_{j\in\mathbb{N}}$, $(x_{n_j})_{j\geqq 1}$, $(x_{n_j})_j$ or simply (x_{n_j}).

Compare this to note 4.1 about simplified notation for families. □

Here are two more examples of sequences.

Example 4.24 (Oscillating sequence)**.** If $x_j := (-1)^j$ $(j \in \mathbb{Z}_{\geqq 0})$, then

$$x_0 = 1, \quad x_2 = -1, \quad x_2 = 1, \quad x_3 = -1, \quad x_4 = 1, \quad x_5 = -1, \dots \quad \square$$

Example 4.25 (Series (summation sequence))**.** Let $s_k := 1 + 2^{-1} + 2^{-2} + \dots + 2^{-k}$ $(k = 1, 2, 3, \dots)$:

$$\begin{aligned} s_1 &= 1, \quad s_2 = 1 + 1/2 = 2 - 1/2, \quad s_3 = 1 + 1/2 + 1/4 = 2 - 1/4, \quad \dots, \\ s_k &= 1 + 1/2 + \dots + 2^{k-1} = 2 - 2^{k-1}; \quad s = 1 + 1/2 + 1/4 + 1/8 + \dots \quad \text{"infinite sum".} \end{aligned}$$

You obtain s_{k+1} from $s_k = 2 - 2^{k-1}$ by cutting the difference 2^{k-1} to the number 2 in half (that would be 2^k) and adding that to s_k. It is intuitively obvious from $s_k = 2 - 2^{k-1}$ that the infinite sum s adds up to 2. Such an infinite sum is called a **series**. [51] □

Remark 4.17. Having defined the family $(x_\imath)_{\imath\in J}$ as the function which maps $\imath \in J$ to $x_\imath$ means that a family distinguishes any two of its members $x_\imath$ and $x_\jmath$ by remembering what their indices are, even if they represent one and the same element of X: Think of "$(x_\imath)_{\imath\in J}$" as an abbreviation for

$$\text{"}\big((\imath, x_\imath)\big)_{\imath\in J}\text{"}. \tag{4.25}$$

[50] $\mathbb{Z}$ is a **linearly ordered set**, also called a **totally ordered set** set in the terminology of def.**??** (Linear orderings) on p.**??**.

[51] The precise definition of a series will be given in ch.10.3 (Function Sequences and Infinite Series) on p.221.

Doing so should also make it much easier to see the equivalence of functions and families: (4.25) looks at its core very much like the graph $\{(\imath, x_\imath) : \imath \in J\}$ of the function $\imath \mapsto x_\imath$. $\square$

Remark 4.18 (Families and sequences can contain duplicates)**.** One of the important properties of sets is that they do not contain any duplicates (see def.2.1 (sets) on p.10). On the other hand, remark 4.17 casually mentions that families, and hence sequences as special kinds of families, can contain duplicates. Let us look at this more closely.

The two sets $A := \{31, 20, 20, 20, 31\}$ and $B := \{20, 31\}$ are equal. On the other hand let $J := \{\alpha, \beta, \pi, \star, Q\}$ and define the family $(w_\imath)_{\imath \in J}$ in B by its associated graph as follows:

$\Gamma := \{(\alpha, 31), (\beta, 20), (\pi, 20), (\star, 20), (Q, 31)\}$, i.e.,, $w_\alpha = 31, w_\beta = 20, , w_\pi = 20, w_\star = 20, w_Q = 31$.

The three occurrences of 20 cannot be distinguished as elements of the set A. In contrast to this the items $(\beta, 20), (\pi, 20), (\star, 20)$ as elements of $\Gamma \subseteq J \times A = J \times B$ [52] are different from each other because two pairs (a, b) and (x, y) are equal only if $x = a$ and $y = b$. $\square$

In contrast to sets, families and sequences allow us to incorporate duplicates.

A family $(x_\imath)_{\imath \in J}$ in X is specified by the function $F : J \to X$ which maps $\imath \in J$ to $F(z) = x_z$. Conversely, let X, Y be nonempty sets and let $f : X \to Y$ be a function with domain X and codomain Y. For $x \in X$ let $f_x := f(x)$. Then f can be written as $(f_x)_{x \in X}$, i.e., as a family in Y with index set X. Thus we have

Proposition 4.4. *The following ways of specifying a function $f : X \to Y$, $x \mapsto f(x)$ are equivalent:*

a. *$f = (X, Y, \Gamma)$ is defined by its graph $\Gamma := \{(x, f(x)) : x \in X\}$.*
b. *f is defined by the following family in Y: $(f(x))_{x \in X}$*

Note that this is one case where we had to explicitly mention the codomain Y in the specification of the family.

There will be a lot more on sequences and series (sequences of sums) in later chapters, but we need to develop more concepts, such as convergence, to continue with this subject. Now let's get back to sets.

4.3 Exercises for Ch.4

4.3.1 Exercises for Functions and Relations

Exercise 4.1.

a. Which of the following is an equivalence relation? a partial ordering? on $\mathbb{R}$?
a1. $xRy \Leftrightarrow x < y$, **a2.** $xRy \Leftrightarrow x \leqq y$, **a3.** $xRy \Leftrightarrow x = y$, **a4.** $xRy \Leftrightarrow x \neq y$.

b. Define $xRy \Leftrightarrow xy > 0$. Is this an equivalence relation on $\mathbb{R}$? on $\mathbb{R}_{\neq 0}$? on $\mathbb{R}_{>0}$? on $\mathbb{R}_{<0}$? $\square$

Exercise 4.2. Injectivity and Surjectivity

- Let $f : \mathbb{R} \to [0, \infty[$; $x \mapsto x^2$.
- Let $g : [0, \infty[\to [0, \infty[$; $x \mapsto x^2$.
  In other words, g is same function as f as far as assigning function values is concerned, but its domain is downsized to $[0, \infty[$.

[52] Be sure to understand that $J \times A = J \times B$!

Answer the following with **true** or **false**.

a. f is surjective **c.** g is surjective
b. f is injective **d.** g is injective

If your answer is **false** then give a specific counterexample. □

Exercise 4.3 (Excercise 4.2 continued)**.** Let $A \subseteq \mathbb{R}$.

Part 1.

- Let $F_1 : A \to [-2, 20[; \quad x \mapsto x^2$.
- Let $F_2 : A \to [2, 20[; \quad x \mapsto x^2$.

What choice of A makes

a. F_1 surjective? **c.** F_2 surjective?
b. F_1 injective? **d.** F_2 injective?

Part 2.

- Let $G_1 : A \to [-2, 20[; \quad x \mapsto \sqrt{x}$.
- Let $G_2 : A \to [2, 20[; \quad x \mapsto \sqrt{x}$.

What choice of A makes

e. G_1 surjective? **g.** G_2 surjective?
f. G_1 injective? **h.** G_2 injective?

Part 3.

- Let $G_3 : A \to [-20, 2[; \quad x \mapsto \sqrt{x}$.
- Let $G_4 : A \to [-20, -2[; \quad x \mapsto \sqrt{x}$.

What choice of A makes

i. G_3 surjective? **k.** G_3 surjective?
j. G_4 injective? **l.** G_4 injective?

For the questions above

- Write **impossible** if no choice of $A \subseteq \mathbb{R}$ exists.
- Write **NAF** for any of $F_1, F_2, G_1, G_2, G_3, G_4$ which does **not define a function**. □

Exercise 4.4. Find $f : X \to Y$ and $A \subseteq X$ such that $f(A^{\complement}) \neq f(A)^{\complement}$. Hint: use $f(x) = x^2$ and choose Y as a **one element only** set (which does not leave you a whole lot of choices for X). See example 4.17 on p.79. □

Exercise 4.5.

a. Prove prop.4.1**a**: The composition of two injective functions is injective.
b. Prove prop.4.1**b**: The composition of two surjective functions is surjective. □

Exercise 4.6. You proved in the previous exercise that
injective $\circ$ injective $=$ injective,
surjective $\circ$ surjective $=$ surjective.

This exercise illustrates that the reverse is not necessarily true.

Find functions $f : \{a\} \to \{b_1, b_2\}$ and $g : \{b_1, b_2\} \to \{a\}$ such that $h := g \circ f : \{a\}$ is bijective but such that it is **not true** that both f, g are injective and it is also **not true** that both f, g are surjective.

Hint: There are not a whole lot of possibilities. Draw possible candidates for f and g in arrow notation as on p.118. You should easily be able to figure out some examples. Think simple and look at example 4.17 on p.79. $\square$

Exercise 4.7. Prove **b** of remark 4.12 on p.86: Let X and Y be two non-empty sets. Then the sets $\{0\} \times X$ and $\{1\} \times Y$ are disjoint. $\square$

Exercise 4.8. Prove **c** of remark 4.12 on p.86: Let X and Y be two non-empty sets.
Then an injection/surjection/bijection $X \to Y$ exists if and only if an injection/surjection/bijection $\{0\} \times X \ \to \ \{1\} \times Y$ exists. $\square$

Exercise 4.9. B/G Project 6.9.:

On $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ we define the relation $\sim$ as follows.

(4.26)
$$(m_1, n_1) \ \sim \ (m_2, n_2) \ \Leftrightarrow m_1 \cdot n_2 = n_1 \cdot m_2.$$

a. Prove that $\sim$ defines an equivalence relation on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$.

Let

(4.27)
$$\mathfrak{Q} \ := \{[(m, n)] \ : \ m, n \in \mathbb{Z} \text{ and } n \neq 0\}$$

be the set of all equivalence classes of $\sim$. We define two binary operations $\oplus$ and $\otimes$ on $\mathfrak{Q}$ as follows;

(4.28)
$$[(m_1, n_1)] \oplus [(m_2, n_2)] \ := [(m_1 n_2 \ + \ m_2 n_1, n_1 n_2)],$$

(4.29)
$$[(m_1, n_1)] \otimes [(m_2, n_2)] \ := [(m_1 m_2, n_1 n_2)]$$

b. Prove that these binary operations are defined consistently: the right hand sides of (4.28) and (4.29) do not depend on the particular choice of elements picked from the sets $[(m_1, n_1)]$ and $[(m_2, n_2)]$. In other words, prove the following:

Let $(p_1, q_1) \sim (m_1, n_1)$ and $(p_2, q_2) \sim (m_2, n_2)$. Then

(4.30)
$$[(m_1 n_2 \ + \ m_2 n_1, n_1 n_2)] \ = [(p_1 q_2 \ + \ p_2 q_1, q_1 q_2)],$$

(4.31)
$$[(m_1 m_2, n_1 n_2)] \ = [(p_1 p_2, q_1 q_2)].$$

or, equivalently, then

(4.32)
$$(m_1 n_2 \ + \ m_2 n_1, n_1 n_2) \ \sim (p_1 q_2 \ + \ p_2 q_1, q_1 q_2),$$

(4.33)
$$(m_1 m_2, n_1 n_2) \ \sim (p_1 p_2, q_1 q_2). \ \square$$

5 More on Sets (Understand this!)

5.1 More on Set Operations (Study this!)

The material in this chapter thematically belongs to ch.2.1 on p.10 but it had to be deferred to this chapter as much of it deals with families of sets, i.e., families $(A_i)_i$

Definition 5.1 (Arbitrary unions and intersections)**.** Let J be a nonempty set and let $(A_i)_{i \in J}$ be a family of sets. We define

$$\text{(5.1)} \qquad \bigcup_{i \in I} A_i \; := \bigcup \big[A_i : i \in I \big] \; := \; \{ x : x \in A_{i_0} \text{ for some } i_0 \in I \},$$

$$\text{(5.2)} \qquad \bigcap_{i \in I} A_i \; := \bigcap \big[A_i : i \in I \big] \; := \; \{ x : x \in A_{i_0} \text{ for each } i_0 \in I \}.$$

We call $\bigcup\limits_{i \in I} A_i$ the **union** and $\bigcap\limits_{i \in I} A_i$ the **intersection** of the family $(A_i)_{i \in J}$

It is convenient to allow unions and intersections for the empty index set $J = \emptyset$. For intersections this requires the existence of a universal set Ω. We define

$$\text{(5.3)} \qquad \bigcup_{i \in \emptyset} A_i \; := \emptyset, \qquad \bigcap_{i \in \emptyset} A_i \; := \; \Omega. \; \square$$

> *Note that any statement concerning arbitrary families of sets such as the definition above covers finite lists $A_1, A_2, \dots, A_n$ of sets ($J = \{1, 2, \dots, n\}$) and also sequences $A_1, A_2, \dots,$ of sets ($J = \mathbb{N}$).*

We give some examples of non-finite unions and intersections. For the first one (prop.5.1) We need to review the behavior of the sequence $x_n = 1/n$.

Remark 5.1. The following definition of the limit of a sequence is known from calculus (see [8] Stewart, J: Single Variable Calculus, 7th edition, ch.11.1) and it will be repeated in ch.8.2 (Convergence and Continuity in $\mathbb{R}$) of this document (see def.8.10 on p.129).

We say that a sequence (x_n) of real numbers **converges** to $a \in \mathbb{R}$ as $n \to \infty$ if for any $\varepsilon \in \mathbb{R}$ (no matter how small) there exists a corresponding (possibly extremely large) $N \in \mathbb{N}$ such that

$$\text{(5.4)} \qquad |x_j - a| < \varepsilon \;\text{ for all }\; j \geqq N.$$

We write

$$\text{(5.5)} \qquad a \;=\; \lim_{n \to \infty} x_n \qquad \text{or} \qquad x_n \to a$$

and we call a the **limit** of the sequence (x_n). $\square$

Remark 5.2. It is known from calculus that $\lim\limits_{n \to \infty} \dfrac{1}{n} \;=\; 0$. According to (5.4) this implies that for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\text{(5.6)} \qquad \frac{1}{n} \;=\; |\frac{1}{n} - 0| < \varepsilon \;\text{ for all }\; n \geqq N. \;\square$$

Proposition 5.1. *For the following note that $[u,v] = \emptyset$ for $u > v$ and $]u,v[= \emptyset$ for $u \geqq v$ (see (2.10) on p.16). Let $a,b \in \mathbb{R}$. Then*

$$[a,b] = \bigcap_{n \in \mathbb{N}} \Big]a - \frac{1}{n}, b + \frac{1}{n}\Big[. \tag{5.7}$$

$$]a,b[= \bigcup_{n \in \mathbb{N}} \Big[a + \frac{1}{n}, b - \frac{1}{n}\Big], \tag{5.8}$$

***A.** Proof of (5.7)*

***Case 1:** We assume that $a > b$. Then $[a,b] = \emptyset$, hence (5.7) is valid if we can show that there exists $N \in \mathbb{N}$ such that $]a - \frac{1}{N}, b + \frac{1}{N}[$ is empty. We do this as follows. Let $\varepsilon := \frac{a-b}{2}$. Then $\varepsilon > 0$. According to (5.6) there exists $N \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$ for all $n \geqq N$; in particular, $\frac{1}{N} < \varepsilon = \frac{a-b}{2}$. Of course the choice of N depends on x. It follows that*

$$\frac{2}{N} < a - b, \quad \textit{hence} \quad b + \frac{1}{N} < a - \frac{1}{N}, \quad \textit{hence} \quad \Big]a - \frac{1}{N}, b + \frac{1}{N}\Big[= \emptyset.$$

***Case 2:** We assume that $a = b$. Then $[a,b] = \{a\}$. Clearly $a \in]a - \frac{1}{n}, b + \frac{1}{n}[$ for all n, hence $\{a\} \subseteq \bigcap_n]a - \frac{1}{n}, b + \frac{1}{n}[$. The proof for case 2 is done if we can show that if $x \neq a$ then there exists $N \in \mathbb{N}$ such that $x \notin]a - \frac{1}{N}, b + \frac{1}{N}[$.*

If $x < a$, let $\varepsilon := a - x > 0$. According to (5.6) there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon = a - x$, i.e., $x < a - \frac{1}{N}$, hence $x \notin]a - \frac{1}{N}, b + \frac{1}{N}[$.

If $x > a$, we can similarly find some$N \in \mathbb{N}$ such that $\frac{1}{N} < x - a$, i.e., $x > a + \frac{1}{N}$, hence $x \notin]a - \frac{1}{N}, b + \frac{1}{N}[$.

***Case 3:** We assume that $a < b$. If $n \in \mathbb{N}$ then $]a - \frac{1}{n}, b + \frac{1}{n}[\supseteq [a,b]$, hence $[a,b] \subseteq \bigcup_{n \in \mathbb{N}}]a - \frac{1}{n}, b + \frac{1}{n}[$.*

We finally show "$\supseteq$". If $x \in \bigcup_{n \in \mathbb{N}}]a - \frac{1}{n}, b + \frac{1}{n}[$ then

$$a - \frac{1}{n} < x < b + \frac{1}{n} \quad \textit{for all } n \in \mathbb{N}. \tag{5.9}$$

The proof is done if we can show that this implies both $x \geqq a$ and $x \leqq b$.

Assume to the contrary that $x < a$. Let $\varepsilon := a - x$. Then $\varepsilon > 0$. According to (5.6) there exists $N \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$ for all $n \geqq N$; in particular, $\frac{1}{N} < \varepsilon$. Of course the choice of N depends on x.

We obtain from $a - x = \varepsilon > \frac{1}{N}$ that $x < a - \frac{1}{N}$. This contradicts (5.9) and we have proved that $x \geqq a$. Demonstrating that $x \leqq b$ is similar. We have proved "$\supseteq$".

***B.** Proof of (5.8): This proof is left as exercise 5.1.* ■

Example 5.1. For any set A we have $A = \bigcup_{a \in A} \{a\}$. According to (5.3) this also is true if $A = \emptyset$. □

The following trivial lemma (a lemma is a "proof subroutine" which is not remarkable on its own but very useful as a reference for other proofs) is useful if you need to prove statements of the form $A \subseteq B$ or $A = B$ for sets A and B. It is a means to simplity the proofs of [1] B/G (Beck/Geoghegan), project 5.12. You must reference this lemma as the "inclusion lemma" when you use it in your homework or exams. Be sure to understand what it means if you choose $J = \{1,2\}$ (draw one or two Venn diagrams).

Lemma 5.1 (Inclusion lemma)**.** *Let J be an arbitrary, non-empty index set. Let U, X_j, Y, Z_j, W $(j \in J)$ be sets such that $U \subseteq X_j \subseteq Y \subseteq Z_j \subseteq W$ for all $j \in J$. Then*

$$U \subseteq \bigcap_{j \in J} X_j \subseteq Y \subseteq \bigcup_{j \in J} Z_j \subseteq W. \tag{5.10}$$

Proof: Note that we need at various places in this proof the existence of some $j_0 \in J$, i.e. the assumption that $J \neq \emptyset$ is essential.

a. *Let $x \in U$. Then $x \in X_j$ for all $j \in J$, hence $x \in \bigcap_{j \in J} X_j$. This proves the first inclusion.*

b. *Now let $x \in \bigcap_{j \in J} X_j$ and $j_0 \in J$. Then $x \in X_j$ for all $j \in J$; in particular, $x \in X_{j_0}$. It follows from $X_{j_0} \subseteq Y$ that $x \in Y$ and we have shown the second inclusion.*

c. *Let $x \in Y$ and $j_0 \in J$. It follows from $Y \subseteq Z_{j_0}$ that $x \in Z_{j_0}$. But then $x \in \{z : z \in Z_j$ for some $j \in J\}$, i.e., $x \in \bigcup_{j \in J} Z_j$. This proves the third inclusion.*

d. *Finally, assume $x \in \bigcup_{j \in J} Z_j$ It follows from the definitions of unions that there exists $j_0 \in J$ such that $x \in Z_{j_0}$. But then $x \in W$ as W contains Z_{j_0}. It follows that $\bigcup_{j \in J} Z_j \subseteq W$. This finishes the proof of the rightmost inclusion.* ■

Definition 5.2 (Disjoint families)**.** Let J be a nonempty set. We call a family of sets $(A_i)_{i \in J}$ a **mutually disjoint family** if any two different sets A_i, A_j have intersection $A_i \cap A_j = \emptyset$, i.e., if any two sets in that family are mutually disjoint. □

Definition 5.3 (Partition)**.** Let $\mathfrak{A} \subseteq 2^{\Omega}$. We call $\mathfrak{A}$ a **partition** or a **partitioning** of Ω if $A \cap B = \emptyset$ for any two $A, B \in \mathfrak{A}$ and $\Omega = \biguplus \Big[A : A \in \mathfrak{A} \Big]$.

We extend this definition to arbitrary families and hence finite collections and sequences of subsets of Ω: Let J be an arbitrary non-empty set, let $(A_j)_{j \in J}$ be a family of subsets of Ω. We call $(A_j)_{j \in J}$ a partition of Ω if it is a mutually disjoint family which satisfies $\Omega = \biguplus \Big[A_j : j \in J \Big]$, i.e., if $\mathfrak{A} := \{A_j : j \in J\}$ is a partition of Ω.
Note that duplicate non-empty sets cannot occur in a disjoint family of sets because otherwise the condition of mutual disjointness does not hold. □

Example 5.2. Here are some examples of partitions.

a. For any set Ω the collection $\{ \{\omega\} : \omega \in \Omega\}$ is a partition of Ω.

b. The empty set is a partition of the empty set and it is its only partition. Do you see that this is a special case of **a**?

c. The set of half open intervals $\{]k, k+1] : k \in \mathbb{Z}\}$ is a partitioning of $\mathbb{R}$.

d. Given is a strictly increasing sequence $n_0 = 0 < n_1 < n_2 < \dots$ of non-negative integers. For $k \in \mathbb{N}$ let $A_k := \{j \in \mathbb{N} : n_{k-1} < j \leqq n_k\}$. Then the set $\{A_k : k \in \mathbb{N}\}$ is a partition of $\mathbb{N}$ (**not** of $\mathbb{Z}_{\geq 0}$!)
□

Theorem 5.1 (De Morgan's Law). *Let there be a universal set Ω (see (2.6) on p.13). Then the following "duality principle" holds for any indexed family $(A_\alpha)_{\alpha\in I}$ of sets:*

(5.11) $$\textbf{a.}\ \Big(\bigcup_\alpha A_\alpha\Big)^{\complement} = \bigcap_\alpha A_\alpha^{\complement} \qquad \textbf{b.}\ \Big(\bigcap_\alpha A_\alpha\Big)^{\complement} = \bigcup_\alpha A_\alpha^{\complement}$$

To put this in words, the complement of an arbitrary union is the intersection of the complements and the complement of an arbitrary intersection is the union of the complements.

Generally speaking the formulas are a consequence of the duality principle for set operations which states that any true statement involving a family of subsets of a universal sets can be converted into its "dual" true statement by replacing all subsets with their complements, all unions with intersections and all intersections with unions.

Proof of De Morgan's law, formula a:

1) *First we prove that $(\bigcup_\alpha A_\alpha)^{\complement} \subseteq \bigcap_\alpha A_\alpha^{\complement}$:*
Assume that $x \in (\bigcup_\alpha A_\alpha)^{\complement}$. Then $x \notin \bigcup_\alpha A_\alpha$ which is the same as saying that x does not belong to any of the A_α. That means that x belongs to each $A_\alpha^{\complement}$ and hence also to the intersection $\bigcap_\alpha A_\alpha^{\complement}$.

2) *Now we prove that $(\bigcup_\alpha A_\alpha)^{\complement} \supseteq \bigcap_\alpha A_\alpha^{\complement}$: Let $x \in \bigcap A_\alpha^{\complement}$. Then x belongs to each of the $A_\alpha^{\complement}$ and hence to none of the A_α. Then it also does not belong to the union of all the A_α and must therefore belong to the complement $(\bigcup_\alpha A_\alpha)^{\complement}$. This completes the proof of formula a. The proof of formula b is not given here because the mechanics are the same.* ■

You should draw the Venn diagrams involving just two sets A_1 and A_2 for both formulas a and b so that you understand the visual representation of De Morgan's law.

Proposition 5.2 (Distributivity of unions and intersections). *Let $(A_i)_{i\in I}$ be an arbitrary family of sets and let B be a set. Then*

(5.12) $$\bigcup_{i\in I}(B\cap A_i) = B \cap \bigcup_{i\in I} A_i,$$

(5.13) $$\bigcap_{i\in I}(B\cup A_i) = B \cup \bigcap_{i\in I} A_i.$$

Proof: We only prove (5.12).
Proof of "$\subseteq$": It follows from $B\cap A_i \subseteq A_i$ for all i that $\bigcup_i(B\cap A_i)\subseteq \bigcup_i A_i$. Moreover, $B\cap A_i\subseteq B$ for all i implies $\bigcup_i(B\cap A_i)\subseteq\bigcup_i B$ which equals B. It follows that $\bigcup_i(B\cap A_i)$ is contained in the intersection $\Big(\bigcup_i A_i\Big)\cap B$.
Proof of "$\supseteq$": Let $x \in B\cap\bigcup_i A_i$. Then $x\in B$ and $x\in A_{i^\star}$ for some $i^\star\in I$, hence $x\in B\cap A_{i^\star}$, hence $x\in\bigcup_i(B\cap A_i)$. ■

Proposition 5.3 (Rewrite unions as disjoint unions). *Let $(A_j)_{j\in\mathbb{N}}$ be a sequence of sets which all are contained within the universal set Ω. For $n\in\mathbb{N}$ let $B_n := \bigcup_{j=1}^{n} A_j = A_1\cup A_2\cup\dots\cup A_n$*

Further, let $C_1 := A_1 = B_1$ and $C_{n+1} := A_{n+1} \setminus B_n$ $(n \in \mathbb{N})$. Then

a. *The sequence $(B_j)_j$ is increasing: $m < n \Rightarrow B_m \subseteq B_n$,*

b. *For each $n \in \mathbb{N}$, $\bigcup_{j=1}^{n} A_j = \bigcup_{j=1}^{n} B_j$,*

c. *The sets C_j are mutually disjoint and $\bigcup_{j=1}^{n} A_j = \bigcup_{j=1}^{n} B_j = \biguplus_{j=1}^{n} C_j$.*

*Proof of **a** and of **b**: Left as exercise 5.2 (p.101).*

*Proof of **c:** Let $1 \leqq j \leqq n$. We note that $C_j \subseteq A_j \subseteq B_j \subseteq B_n$ and obtain*

$$C_j \cap C_{n+1} \subseteq B_n \cap C_{n+1} = B_n \cap (A_{n+1} \setminus B_n) = B_n \cap (A_{n+1} \cap B_n^{\complement}) = A_{n+1} \cap (B_n \cap B_n^{\complement}) = \emptyset.$$

We have proved that for any $j, k \in \mathbb{N}$ such that $j < k$ the sets C_j and C_k have empty intersection (we replaced $n+1$ with k) and it follows that the entire sequence of sets C_j is disjoint. ■

5.2 Cartesian Products of more than Two Sets

Remark 5.3 (Associativity of cartesian products)**.** Assume we have three sets A, B and C. We can then look at

$$\begin{aligned}(A \times B) \times C &= \{((a,b),c) : a \in A, b \in B, c \in C\}\\ A \times (B \times C) &= \{(a,(b,c)) : a \in A, b \in B, c \in C\}\end{aligned}$$

The mapping

$$F : (A \times B) \times C \to A \times (B \times C), \qquad \big((a,b),c\big) \mapsto \big(a,(b,c)\big)$$

is bijective because it has the mapping

$$G : A \times (B \times C) \to (A \times B) \times C, \qquad \big(a,(b,c)\big) \mapsto \big((a,b),c\big)$$

as an inverse. For both $(A \times B) \times C$ and $A \times (B \times C)$ there are bijections to the set $\{(a,b,c) : a \in A, b \in B, c \in C\}$ of all triplets (a,b,c): the obvious bijections would be $(a,b,c) \mapsto \big((a,b),c\big)$ and $(a,b,c) \mapsto \big((a,b),c\big)$. □

This remark leads us to the following definition:

Definition 5.4 (Cartesian Product of three or more sets)**.** The **cartesian product** of three sets A, B and C is defined as

$$A \times B \times C := \{(a,b,c) : a \in A, b \in B, c \in C\}$$

i.e., it consists of all pairs (a,b,c) with $a \in A$, $b \in B$ and $c \in C$.

More generally, for N sets $X_1, X_2, X_3, \dots, X_N$ $(N \in \mathbb{N})$, we define the **cartesian product** as [53]

$$X_1 \times X_2 \times X_3 \times \dots \times X_N \;:=\; \{(x_1, x_2, \dots, x_N) : x_j \in X_j \text{ for all } 1 \leqq j \leqq N\}$$

Two elements $(x_1, x_2, \dots, x_N)$ and $(y_1, y_2, \dots, y_N)$ of $X_1 \times X_2 \times X_3 \times \dots \times X_N$ are called **equal** if and only if $x_j = y_j$ for all j such that $1 \leqq j \leqq N$. In this case we write $(x_1, x_2, \dots, x_N) = (y_1, y_2, \dots, y_N)$.

As a shorthand, we abbreviate $X^N := \underbrace{X \times X \times + \dots \times X}_{N \text{ times}}$. ☐

Example 5.3 (N–dimensional coordinates)**.** Here is the most important example of a cartesian product of N sets. Let $X_1 = X_2 = \dots = X_N = \mathbb{R}$. Then

$$\mathbb{R}^N \;=\; \{(x_1, x_2, \dots, x_N) : x_j \in \mathbb{R} \text{ for } 1 \leqq j \leqq N\}$$

is the set of points in N–dimensional space. You may not be familiar with what those are unless $N = 2$ (see example 4.1 above) or $N = 3$.

In the 3–dimensional case it is customary to write (x, y, z) rather than (x_1, x_2, x_3). Each such triplet of real numbers represents a point in (ordinary 3–dimensional) space and we speak of its x–coordinate, y–coordinate and z–coordinate.

For the sake of completeness: If $N = 1$, the item $(x) \in \mathbb{R}^1$ (where $x \in \mathbb{R}$; observe the parentheses around x) is considered the same as the real number x. In other words, we "identify" $\mathbb{R}^1$ with $\mathbb{R}$. Such a "one–dimensional point" is simply a point on the x–axis.

A short word on vectors and coordinates: For $N \leqq 3$ you can visualize the following: Given a point x on the x–axis or in the plane or in 3–dimensional space, there is a unique arrow that starts at the point whose coordinates are all zero (the "origin") and ends at the location marked by the point x. Such an arrow is customarily called a vector.

Because it makes sense in dimensions $1, 2, 3$, an N**–tuple** $(x_1, x_2, \dots, x_N)$ is called a vector of dimension N. You will read more about this in ch.9 about vectors and vector spaces on page 151.

This is worth while repeating: We can uniquely identify each $x \in \mathbb{R}^N$ with the corresponding vector: an arrow that starts in $(\underbrace{0, 0, \dots, 0}_{N \text{times}})$ and ends in x.

More will be said about n-dimensional space in section 9, p.151 on Vectors and vector spaces. ☐

Example 5.4 (Parallelepipeds)**.** Let $a_1 < b_1, a_2 < b_2, a_3 < b_3$ be real numbers. Then

$$[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \;=\; \{(x, y, z) : a_1 \leqq x \leqq b_1, a_2 \leqq y \leqq b_2, a_3 \leqq z \leqq b_3\}$$

[53] If $N > 3$ there are many ways to group the factors of a cartesian product. For $N = 4$ there already are 3 times as many possibilities as for $N = 3$:

$$X_1 \times (X_2 \times X_3 \times X_4), \quad (X_1 \times X_2) \times (X_3 \times X_4), \quad X_1 \times (X_2 \times X_3 \times X_4),$$

Actually proving that we can group the sets with parentheses any way we like is very tedious and will not be done in this document.

is the **parallelepiped** (box or quad parallel to the coordinate axes) with sides $[a_1, b_1], [a_2, b_2]$ and $[a_3, b_3]$. This generalizes in an obvious manner to N dimensions:

Let $N \in \mathbb{N}$ and $a_j < b_j$ ($j \in \mathbb{N}, j \leqq N, a_j, b_j \in \mathbb{R}$). Then

$$[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_N, b_N] \;=\; \{(x_1, x_2, \dots, x_N) : a_j \leqq x_j \leqq b_j, j \in \mathbb{N}, j \leqq N\}$$

is the parallelepiped with sides $[a_1, b_1], \dots, [a_N, b_N]$. $\square$

We now introduce cartesian products of an entire family of sets $(X_i)_{i \in I}$.

Definition 5.5 (Cartesian Product of a family of sets)**.** Let I be an arbitrary, non–empty set (the index set). Let $(X_i)_{i \in I}$ be a family of non–empty sets X_i. The **cartesian product** of the family $(X_i)_{i \in I}$ is the set

$$\prod_{i \in I} X_i \;:=\; \Big(\prod X_i\Big)_{i \in I} \;:=\; \{(x_i)_{i \in I} : x_k \in X_k \; \forall k \in I\}$$

of all familes $(x_i)_{i \in I}$ each of whose members x_j belongs to the corresponding set X_j. The "$\prod$" is the greek "upper case" letter "Pi" (whose lower case incarnation "π" you are probably more familiar with).

Two elements $(x_i)_{i \in I}$ and $(y_k)_{k \in I}$ of $\prod\limits_{i \in I} X_i$ are called **equal** if and only if $x_j = y_j$ for all $j \in I$. In this case we write $(x_i)_{i \in I} = (y_k)_{k \in I}$. [54]

If all sets X_i are equal to one and the same set X, we also write $X^I := \prod\limits_{i \in I} X := \prod\limits_{i \in I} X_i$. $\square$

Remark 5.4. We note that each element $(y_x)_{x \in X}$ of the cartesian product Y^X is the function

$$y(\cdot) : X \to Y, \qquad x \mapsto y_x$$

(see def.4.17 (indexed families) and the subsequent remarks concerining the equivalence of functions and families). In other words,

$$Y^X \;=\; \{f : f \text{ is a function with domain } X \text{ and codomain } Y\}. \quad \square \tag{5.14}$$

5.3 Exercises for Ch.5

Exercise 5.1. Prove (5.8) of prop.5.1 on p.96: Let $a, b \in \mathbb{R}$. Then $]a, b[= \bigcup\limits_{n \in \mathbb{N}} [a + 1/n, b - 1/n]$. Adapt the proof of (5.7) but note that this one is simpler. There are only two cases to worry about: $a \geqq b$ (very easy!) vs $a < b$.

Exercise 5.2. Prove **a** and **b** of prop.5.3 (Rewrite unions as disjoint unions) on p.98:

Let $(A_j)_{j \in \mathbb{N}}$ such that $A_j \subseteq \Omega$ for all $j \in \mathbb{N}$. For $n \in \mathbb{N}$ let $B_n := \bigcup\limits_{j=1}^{n} A_j \;=\; A_1 \cup A_2 \cup \cdots \cup A_n$

Further, let $C_1 := A_1 = B_1$ and $C_{n+1} := A_{n+1} \setminus B_n$ ($n \in \mathbb{N}$). Then

[54] In other words, if and only if those two families are equal in the sense of def.4.18 on p.90.

a. The sequence $(B_j)_j$ is increasing: $m < n \Rightarrow B_m \subseteq B_n$,

b. For each $n \in \mathbb{N}$, $\displaystyle\bigcup_{j=1}^{n} A_j \;=\; \bigcup_{j=1}^{n} B_j$. $\square$

6 Sets and Functions, Direct and Indirect Images (Study this!)

6.1 Direct Images and Indirect Images (Preimages) of a Function

Definition 6.1. Let X, Y be two non-empty sets and $f : X \to Y$ be an arbitrary function with domain X and codomain Y. Let $A \subseteq X$ and $B \subseteq Y$. Let

$$f(A) := \{f(x) : x \in A\}, \tag{6.1}$$

$$f^{-1}(B) := \{x \in X : f(x) \in B\}. \tag{6.2}$$

We call $f(A)$ the **direct image** [55] of A under f and we call We call $f^{-1}(B)$ the **indirect image** or **preimage** of B under f. $\square$

Notational conveniences:

If we have a set that is written as $\{\dots\}$ then we may write $f\{\dots\}$ instead of $f(\{\dots\})$ and $f^{-1}\{\dots\}$ instead of $f^{-1}(\{\dots\})$. Specifically for $x \in X$ and $y \in Y$ we get $f^{-1}\{x\}$ and $f^{-1}\{y\}$. Many mathematicians will write $f^{-1}(y)$ instead of $f^{-1}\{y\}$ but this writer sees no advantages doing so whatsover. There seemingly are no savings with respect to time or space for writing that alternate form but we are confounding two entirely separate items: a subset $f^{-1}\{y\}$ of X v.s. the function value $f^{-1}(y)$ of $y \in Y$ which is an element of X. We can talk about the latter only in case that the inverse function f^{-1} of f exists.

In measure theory and probability theory the following notation is also very common: $\{f \in B\}$ rather than $f^{-1}(B)$ and $\{f = y\}$ rather than $f^{-1}\{y\}$

Let $a < b \in \mathbb{R}$. We write $\{a \leqq f \leqq b\}$ for $f^{-1}([a,b])$, $\{a < f < b\}$ for $f^{-1}(]a,b[)$, $\{a \leqq f < b\}$ for $f^{-1}([a,b[)$ and $\{a < f \leqq b\}$ for $f^{-1}(]a,b])$, $\{f \leqq b\}$ for $f^{-1}(]-\infty,b])$, etc.

Example 6.1 (Direct images)**.** Let $f : \mathbb{R} \to \mathbb{R}$; $\quad f(x) = x^2$.

- **a.** $f(]-4,-2[) = \{\, x^2 : x \in]-4,-2[\,\} = \{\, x^2 : -4 < x < -2 \,\} =]4,16[$.
- **b.** $f([1,2]) = \{\, x^2 : x \in [1,2] \,\} = \{\, x^2 : 1 \leqq x \leqq 2 \,\} = [1,4]$.
- **c.** $f([5,6]) = \{\, x^2 : x \in [5,6] \,\} = \{\, x^2 : 5 \leqq x \leqq 6 \,\} = [25,36]$.
- **d.** $f(]-4,-2[\cup [1,2] \cup [5,6]) = \{\, x^2 : x \in]-4,-2[$ or $x \in [1,2]$ or $x \in [5,6] \,\}$
  $=]4,16[\cup [1,4] \cup [25,36] = [1,16[\cup [25,36]$. $\square$

Example 6.2 (Direct images)**.** Let $f : \mathbb{R} \to \mathbb{R}$; $\quad f(x) = x^2$.

- **a.** $f(]-4,2[) = \{\, x^2 : x \in]-4,2[\,\} = \{\, x^2 : -4 < x < 2 \,\} =]4,16[$.
- **b.** $f([1,3]) = \{\, x^2 : x \in [1,3] \,\} = \{\, x^2 : 1 \leqq x \leqq 3 \,\} = [1,9]$.
- **c.** $f(]-4,2[\cap [1,3]) = \{\, x^2 : x \in]-4,2[$ and $x \in [1,3] \,\} = \{\, x^2 : 1 \leqq x < 2 \,\} = [1,4[$. $\square$

And here are the results for the preimages of the same sets with respect to the same function $x \mapsto x^2$.

Example 6.3 (Preimages)**.** Let $f : \mathbb{R} \to \mathbb{R}$; $\quad f(x) = x^2$.

[55] The range $f(X)$ of f (see (4.9) on p.75 is a special case of a direct image.

a. $f^{-1}(]-4,-2[) = \{ x \in \mathbb{R} : x^2 \in]-4,-2[\} = \{ -4 < f < -2 \} = \emptyset.$
b. $f^{-1}([1,2]) = \{ x \in \mathbb{R} : x^2 \in [1,2] \} = \{ 1 \leqq f \leqq 2 \} = [-\sqrt{2},-1] \cup [1,\sqrt{2}].$
c. $f^{-1}([5,6]) = \{ x \in \mathbb{R} : x^2 \in [5,6] \} = \{ 5 \leqq f \leqq 6 \} = [-\sqrt{6},-\sqrt{5}] \cup [\sqrt{5},\sqrt{6}].$
d. $f^{-1}(]-4,-2[\cup [1,2] \cup [5,6]) = \{ x \in \mathbb{R} : x^2 \in]-4,-2[\text{ or } x^2 \in [1,2] \text{ or } x^2 \in [5,6] \}$
$= [-\sqrt{2},-1] \cup [1,\sqrt{2}] \cup [-\sqrt{6},-\sqrt{5}] \cup [\sqrt{5},\sqrt{6}].$ □

Example 6.4 (Preimages)**.** Let $f : \mathbb{R} \to \mathbb{R}; \quad f(x) = x^2$.

a. $f^{-1}(]-4,2[) = \{ x \in \mathbb{R} : x^2 \in]-4,2[\} = \{ x \in \mathbb{R} : -4 < x^2 < 2 \} =]-2,2[.$
b. $f^{-1}([1,3]) = \{ x \in \mathbb{R} : x^2 \in [1,3] \} = \{ x \in \mathbb{R} : 1 \leqq x^2 \leqq 3 \} = [-\sqrt{3},1] \cup [1,\sqrt{3}].$
c. $f^{-1}(]-4,2[\cap [1,3]) = \{ x \in \mathbb{R} : x^2 \in]-4,2[\text{ and } x^2 \in [1,3] \}$
$= \{ x \in \mathbb{R} : 1 \leqq x^2 < 2 \} =]-\sqrt{2},-1] \cup [1,\sqrt{2}[.$ □

> *Unless stated otherwise, X, Y and f are as defined above for the remainder of this chapter: $f : X \to Y$ is a function with domain X and codomain Y.*

Proposition 6.1. *Some simple properties:*

(6.3) $$f(\emptyset) = f^{-1}(\emptyset) = \emptyset$$
(6.4) $$A_1 \subseteq A_2 \subseteq X \Rightarrow f(A_1) \subseteq f(A_2)$$
(6.5) $$B_1 \subseteq B_2 \subseteq Y \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$$
(6.6) $$x \in X \Rightarrow f(\{x\}) = \{f(x)\}$$
(6.7) $$f(X) = Y \quad \Leftrightarrow \quad f \text{ is surjective}$$
(6.8) $$f^{-1}(Y) = X \quad \text{always!}$$

Proof: Left as an exercise. ■

Proposition 6.2 (f^{-1} is compatible with all basic set ops)**.** *In the following we assume that J is an arbitrary index set, and that $B \subseteq Y$, $B_j \subseteq Y$ for all j. Then*

(6.9) $$f^{-1}(\bigcap_{j \in J} B_j) = \bigcap_{j \in J} f^{-1}(B_j)$$
(6.10) $$f^{-1}(\bigcup_{j \in J} B_j) = \bigcup_{j \in J} f^{-1}(B_j)$$
(6.11) $$f^{-1}(B^{\complement}) = f^{-1}(B)^{\complement}$$
(6.12) $$f^{-1}(B_1 \setminus B_2) = f^{-1}(B_1) \setminus f^{-1}(B_2)$$
(6.13) $$f^{-1}(B_1 \Delta B_2) = f^{-1}(B_1) \Delta f^{-1}(B_2)$$

Proof of (6.9): Let $x \in X$. Then

(6.14)
$$\begin{aligned} x \in f^{-1}(\bigcap_{j \in J} B_j) &\Leftrightarrow f(x) \in \bigcap_{j \in J} B_j \quad (\textit{def preimage}) \\ &\Leftrightarrow \forall j \; f(x) \in B_j \quad (\textit{def} \cap) \\ &\Leftrightarrow \forall j \; x \in f^{-1}(B_j) \quad (\textit{def preimage}) \\ &\Leftrightarrow x \in \bigcap_{j \in J} f^{-1}(B_j) \quad (\textit{def} \cap) \end{aligned}$$

Proof of (6.10): Let $x \in X$. Then

$$
\begin{aligned}
x \in f^{-1}(\bigcup_{j\in J} B_j) \;&\Leftrightarrow\; f(x) \in \bigcup_{j\in J} B_j \quad (\textit{def preimage}) \\
&\Leftrightarrow\; \exists j_0 :\; f(x) \in B_{j_0} \quad (\textit{def} \cup) \\
&\Leftrightarrow\; \exists j_0 : x \in f^{-1}(B_{j_0}) \quad (\textit{def preimage}) \\
&\Leftrightarrow\; x \in \bigcup_{j\in J} f^{-1}(B_j) \quad (\textit{def} \cup)
\end{aligned} \tag{6.15}
$$

Proof of (6.11): Let $x \in X$. Then

$$
\begin{aligned}
x \in f^{-1}(B^{\complement}) \;&\Leftrightarrow\; f(x) \in B^{\complement} \quad (\textit{def preimage}) \\
&\Leftrightarrow\; f(x) \notin B \quad (\textit{def } (\cdot)\complement) \\
&\Leftrightarrow\; x \notin f^{-1}(B) \quad (\textit{def preimage}) \\
&\Leftrightarrow\; x \in f^{-1}(B)^{\complement} \quad ((\cdot)\complement)
\end{aligned} \tag{6.16}
$$

Proof of (6.12): Let $x \in X$. Then

$$
\begin{aligned}
x \in f^{-1}(B_1 \setminus B_2) \;&\Leftrightarrow\; x \in f^{-1}(B_1 \cap B_2^{\complement}) \quad (\textit{def} \setminus) \\
&\Leftrightarrow\; x \in f^{-1}(B_1) \cap f^{-1}(B_2^{\complement}) \quad (\textit{see } (6.9)) \\
&\Leftrightarrow\; x \in f^{-1}(B_1) \cap f^{-1}(B_2)^{\complement} \quad (\textit{see } (6.11)) \\
&\Leftrightarrow\; x \in f^{-1}(B_1) \setminus f^{-1}(B_2) \quad (\textit{def} \setminus)
\end{aligned} \tag{6.17}
$$

Proof of (6.13): This follows from $B_1 \Delta B_2 = (B_1 \setminus B_2) \cup (B_2 \setminus B_1)$ and (6.10) *and* (6.12). ■

Proposition 6.3 (Properties of the direct image)**.** *In the following we assume that J is an arbitrary index set, and that $A \subseteq X$, $A_j \subseteq X$ for all j. Then*

$$
f(\bigcap_{j\in J} A_j) \subseteq \bigcap_{j\in J} f(A_j) \tag{6.18}
$$

$$
f(\bigcup_{j\in J} A_j) = \bigcup_{j\in J} f(A_j) \tag{6.19}
$$

Proof of (6.18): This follows from the monotonicity of the direct image (see 6.4):

$$
\begin{aligned}
\bigcap_{j\in J} A_j \subseteq A_i \;\forall i \in J \;&\Rightarrow f(\bigcap_{j\in J} A_j) \subseteq f(A_i) \;\forall i \in J \\
&\Rightarrow f(\bigcap_{j\in J} A_j) \subseteq \bigcap_{i\in J} f(A_i) \quad (\textit{def} \cap)
\end{aligned}
$$

First proof of (6.19)) - "Expert proof":

$$y \in f(\bigcup_{j\in J} A_j) \;\Leftrightarrow\; \exists\, x \in X:\; f(x) = y \text{ and } x \in \bigcup_{j\in J} A_j \quad (\text{def } f(A)) \tag{6.20}$$

$$\Leftrightarrow\; \exists\, x \in X \text{ and } j_0 \in J:\; f(x) = y \text{ and } x \in A_{j_0} \quad (\text{def} \cup) \tag{6.21}$$

$$\Leftrightarrow\; \exists\, x \in X \text{ and } j_0 \in J:\; f(x) = y \text{ and } f(x) \in f(A_{j_0}) \quad (\text{def } f(A)) \tag{6.22}$$

$$\Leftrightarrow\; \exists\, j_0 \in J:\; y \in f(A_{j_0}) \quad (\text{def } f(A)) \tag{6.23}$$

$$\Leftrightarrow\; y \in \bigcup_{j\in J} f(A_j) \quad (\text{def} \cup) \tag{6.24}$$

Alternate proof of (6.19) *- Proving each inclusion separately. Unless you have a lot of practice reading and writing proofs whose subject is the equality of two sets you should write your proof the following way:*

A. Proof of "⊆":

$$y \in f(\bigcup_{j\in J} A_j) \;\Rightarrow \exists\, x \in X:\; f(x) = y \text{ and } x \in \bigcup_{j\in J} A_j \quad (\text{def } f(A)) \tag{6.25}$$

$$\Rightarrow \exists\, j_0 \in J:\; f(x) = y \text{ and } x \in A_{j_0} \quad (\text{def} \cup) \tag{6.26}$$

$$\Rightarrow y = f(x) \in f(A_{j_0}) (\text{def } f(A)) \tag{6.27}$$

$$\Rightarrow y \in \bigcup_{j\in J} f(A_j) \quad (\text{def} \cup) \tag{6.28}$$

B. Proof of "⊇":

This follows from the monotonicity of $A \mapsto f(A)$*:*

$$A_i \subseteq \bigcup_{j\in J} A_j \;\forall\, i \in J \;\Rightarrow f(A_i) \subseteq f\big(\bigcup_{j\in J} A_j\big) \;\forall\, i \in J \tag{6.29}$$

$$\Rightarrow \bigcup_{i\in J} f(A_i) \subseteq f\big(\bigcup_{j\in J} A_j\big) \;\forall\, i \in J \quad (\text{def} \cup) \tag{6.30}$$

■

You see that the "elementary" proof is barely longer than the first one, but it is so much easier to understand!

Remark 6.1. In general you will not have equality in (6.18). Counterexample: $f(x) = x^2$ with domain $\mathbb{R}$: Let $A_1; =] - \infty, 0]$ and $A_2; = [0, \infty[$. Then $A_1 \cap A_2 = \{0\}$, hence $f(A_1 \cap A_2) = f(\{0\}) = \{0\}$. On the other hand, $f(A_1) = f(A_2) = [0, \infty]$, hence $f(A_1) \cap f(A_2) = [0, \infty]$. Clearly, $\{0\} \subsetneq [0, \infty]$. □

Proposition 6.4 (Direct images and preimages of function composition)**.** *Let* X, Y, Z *be an arbitrary, non–empty sets. Let* $f : X \to Y$ *and* $g : Y \to Z$ *and let* $W \subseteq Z$*. Then*

$$(g \circ f)(U) = g\big(f(U)\big) \text{ for all } U \subseteq X. \tag{6.31}$$

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}, \text{ i.e., } (g \circ f)^{-1}(W) = f^{-1}\big(g^{-1}(W)\big) \text{ for all } W \subseteq Z. \tag{6.32}$$

Proof of (6.31)*: Left as exercise* 6.2*.*

Proof of (6.32)*:*
a. "$\subseteq$": Let $W \subseteq Z$ and $x \in (g \circ f)^{-1}(W)$. Then $g\big(f(x)\big) = (g \circ f)(x) \in W$, hence $f(x) \in g^{-1}(W)$. But then $x \in f^{-1}\big(g^{-1}(W)\big)$. This proves "$\subseteq$".

b. "$\supseteq$": Let $W \subseteq Z$, $h := g \circ f$, and $x \in f^{-1}\big(g^{-1}(W)\big)$. Then $f(x) \in g^{-1}(W)$, hence $h(x) = g(f(x)) \in W$, hence $x \in h^{-1}(W) = (g \circ f)^{-1}(W)$. This proves "$\supseteq$". ■

Proposition 6.5 (Indirect image and fibers of f)**.** *Let X, Y be non–empty sets and let $f : X \to Y$ be a function. We define on the domain X a relation "$\sim$" as follows:*

$$x_1 \sim x_2 \Leftrightarrow f(x_1) = f(x_2). \tag{6.33}$$

***a.** "$\sim$" is an equivalence relation. Its equivalence classes, which we denote by $[x]_f$,* [56] *are*

$$[x]_f = \{a \in X : f(a) = f(x)\} = f^{-1}\{f(x)\}. \quad (x \in X) \tag{6.34}$$

***b.** If $A \subseteq X$ then*

$$f^{-1}(f(A)) = \bigcup_{a \in A} [a]_f. \tag{6.35}$$

*Proof of (*6.34*): The equation on the left is nothing but the definition of the equivalence classes generated by an equivalence relation, the equation on the right follows from the definition of preimages.*

*Proof of (*6.35*):*

As $f(A) = f(\bigcup_{x \in A}\{x\}) = \bigcup_{x \in A}\{f(x)\}$ (see 6.19*), it follows that*

$$f^{-1}(f(A)) = f^{-1}\big(\bigcup_{x \in A}\{f(x)\}\big) \tag{6.36}$$

$$= \bigcup_{x \in A} f^{-1}\{f(x)\} \quad \text{(see 6.10)} \tag{6.37}$$

$$= \bigcup_{x \in A} [x]_f \quad \text{(see 6.34)} \tag{6.38}$$

■

Corollary 6.1.

$$\text{If } A \subseteq X \quad \text{then} \quad f^{-1}(f(A)) \supseteq A. \tag{6.39}$$

Proof: It follows from $x \sim x$ for all $x \in X$ that $x \in [x]_f$, i.e., $\{x\} \subseteq [x]_f$ for all $x \in X$. But then

$$A = \bigcup_{x \in A}\{x\} \subseteq \bigcup_{x \in A}[x]_f = f^{-1}(f(A)), \tag{6.40}$$

*where the last equation holds because of (*6.35*).* ■

[56] $[x]_f$ is called the **fiber over** $f(x)$ of the function f.

Proposition 6.6.

$$\text{If } B \subseteq Y \quad \text{then} \quad f(f^{-1}(B)) \;=\; B \cap f(X). \tag{6.41}$$

Proof of "$\subseteq$":

Let $y \in f(f^{-1}(B))$. There exists $x_0 \in f^{-1}(B)$ such that $f(x_0) = y$ (def direct image). We have

a. $x_0 \in f^{-1}(B) \Rightarrow f(x_0) \in B$ *(def. of preimage)*

b. *Of course $x_0 \in X$. Hence $f(x_0) \in f(X)$.*

***a** and **b** together imply that $y = f(x_0) \in B \cap f(X)$.*

Proof of "$\supseteq$":

Let $y \in f(X) \cap B$. We must prove that $y \in f\big(f^{-1}(B)\big)$.

It follows from $y \in f(X)$ that there exists $x_0 \in X$ such that $y = f(x_0)$.

Because $f(x_0) \in B$, we conclude that $x_0 \in f^{-1}(B)$ (def preimage), hence $y = f(x_0) \in f\big(f^{-1}(B)\big)$.

We have shown that if $y \in f(X)$ and $y \in B$ then $y \in f(f^{-1}(B))$. The proof of "$\supseteq$" is completed. ■

Remark 6.2. Be sure to understand how the assumption $y \in f(X)$ was used. □

Corollary 6.2.

$$\text{If } B \subseteq Y \quad \text{then} \quad f(f^{-1}(B)) \;\subseteq\; B. \tag{6.42}$$

Trivial as $f(f^{-1}(B)) \;=\; B \cap f(X) \;\subseteq\; B.$ ■

Proposition 6.7.

a. *Let $A \subseteq X$. If $f : X \to Y$ is injective then $f^{-1}(f(A)) = A$.*

b. *Let $B \subseteq Y$. If $f : X \to Y$ is surjective then $f\big(f^{-1}(B)\big) = B$.*

c. *Let $A \subseteq X$ and $B \subseteq Y$. If $f : X \to Y$ is injective and if $B = f(A)$ then $f^{-1}(B) = A$.*

d. *Let $A \subseteq X$ and $B \subseteq Y$. If $f : X \to Y$ is surective and if $f^{-1}(B) = A$ then $B = f(A)$.*

e. *Let $A \subseteq X$ and $B \subseteq Y$. If $f : X \to Y$ is bijective then $B = f(A) \;\Leftrightarrow\; f^{-1}(B) = A$.*

Proof: Left as exercise 6.3.

Proposition 6.8. *In the following we assume that J is an arbitrary index set, and that $A \subseteq X$, $A_j \subseteq X$ for all j.*
Let $f : X \to Y$ be bijective. Then the following all are true:

$$f\Big(\bigcap_{j\in J} A_j\Big) \;=\; \bigcap_{j\in J} f(A_j) \tag{6.43}$$

$$f\Big(\bigcup_{j\in J} A_j\Big) \;=\; \bigcup_{j\in J} f(A_j) \tag{6.44}$$

$$f(A^{\complement}) \;=\; f(A)^{\complement} \tag{6.45}$$

$$f(A_1 \setminus A_2) \;=\; f(A_1) \setminus f(A_2) \tag{6.46}$$

$$f(A_1 \Delta A_2) \;=\; f(A_1) \Delta f(A_2) \tag{6.47}$$

Proof: Left as exercise 6.4.

Proposition 6.9. *Let J be an arbitrary non–empty index set and let $(A_j)_{j \in J}$ be a partition of X, i.e., if $i \neq j$ then $A_i \cap A_j = \emptyset$ and $X = \biguplus_j A_j$. Assume further that none of the A_j are enpty. For $j \in J$ let $B_j := f(A_j)$. Then*

- ***a.*** *$(B_j)_{j \in J}$ is a partition of Y.*
- ***b.*** *For $j \in J$ we look at the restriction $f\big|_{A_j} : A_j \to Y$ to A_j. Then $f\big|_{A_j}(A_j) = B_j$ and the function*

$$f_j : A_j \to B_j, \qquad x \mapsto f_j(x) := f\big|_{A_j}(x) = f(x)$$

is a bijection.

Proof: Left as exercise 6.5.

Corollary 6.3. *Let $f : X \to Y$ be bijective. Let $A \subset X$, $A \neq \emptyset$ (strict inclusion, so $A^{\complement} \neq \emptyset$). Then both*

$$f_A : A \to f(A), \quad x \to f(x) \qquad \text{and} \quad f_{A^{\complement}} : A^{\complement} \to f(A^{\complement})$$

are bijections.

Proof: This follows from prop.6.9, applied to $J = \{1,2\}, A_1 = A, A_2 = A^{\complement}$. ■

Corollary 6.4. *Let $f : X \to Y$ be bijective. Let $a \subset X$ and assume that $X \neq \{a\}$. Then*

$$\tilde{f} : X \setminus \{a\} \to Y \setminus \{f(a)\}, \quad x \to f(x)$$

also is bijective. [57]

Proof: This follows from 6.4 applied to $A = \{a\}$ and the fact that $f(\{a\}) = \{f(a)\}$. ■

The following two propositions allow you to replace bijective and surjective functions with more suitable ones that inherit bijectivity or surjectivity. This will come in handy in ch.7.1 on p.114 when we prove propositions concerning cardinality.

The first proposition shows how to preserve bijectivity if two function values need to be switched around.

Proposition 6.10. *Let $X, Y \neq \emptyset$, let $f : X \to Y$ be bijective and let $x_1, x_2 \in X$. Let*

$$g(x) := \begin{cases} f(x_2) & \text{if } x = x_1, \\ f(x_1) & \text{if } x = x_2, \\ f(x) & \text{if } x \neq x_1, x_2. \end{cases} \tag{6.48}$$

(In other words, we swap two function arguments). Then $g : X \to Y$ also is bijective.

Proof: *Let $y_1 := f(x_1)$ and $y_2 := f(x_2)$. Let $f^{-1} : Y \to X$ be the inverse function of f and define $G : Y \to X$ as follows*

$$G(y) := \begin{cases} f^{-1}(y_2) & \text{if } y = y_1, \\ f^{-1}(y_1) & \text{if } y = y_2, \\ f^{-1}(y) & \text{if } y \neq y_1, y_2. \end{cases} \tag{6.49}$$

[57] This is B/G [1] prop.13.2.

We show that G satisfies $G \circ g = id_X$ and $g \circ G = id_Y$, i.e., g has G as its inverse. This suffices to prove bijectivity of g.

$$\begin{aligned} y \neq y_1, y_2 \quad \Rightarrow \quad g \circ G(y) &= g(f^{-1}(y)) = f(f^{-1}(y)) = y \;\text{ as } f^{-1}(y) \neq x_1, x_2, \\ g \circ G(y_1) &= g(f^{-1}(y_2)) = g(x_2) = f(x_1) = y_1 \;\text{ as } f^{-1}(y_2) = x_2, \\ g \circ G(y_2) &= g(f^{-1}(y_1)) = g(x_1) = f(x_2) = y_2 \;\text{ as } f^{-1}(y_1) = x_1. \end{aligned}$$

Further,

$$\begin{aligned} x \neq x_1, x_2 \quad \Rightarrow \quad G \circ g(x) &= G(f(x)) = f^{-1}f(x)) = y \;\text{ as } f(x) \neq y_1, y_2, \\ G \circ g(x_1) &= G(f(x_2)) = G(y_2) = f^{-1}(y_1)) = x_1 \;\text{ as } f(x_1) = y_1, \\ G \circ g(x_2) &= G(f(x_1)) = G(y_1) = f^{-1}(y_2)) = x_2 \;\text{ as } f(x_2) = y_2. \end{aligned}$$

We have proved that g has an inverse, the function G. ■

Note that the validity of $G \circ g = id_X$ and $g \circ G = id_Y$ is obvious without the use of any formalism: g differs from f only in that it switches around the function values $f(x_1)$ and $f(x_2)$. and G differs from f^{-1} only in that this switch is reverted.

A more general version of the above shows how to preserve surjectivity if two function values need to be switched around.

Proposition 6.11. *Let $X, Y \neq \emptyset$ and assume that Y contains at least two elements y_1 and y_2. Let $f : X \to Y$ be surjective.*

Let $A_1 := f^{-1}\{y_1\}$, $A_2 := f^{-1}\{y_2\}$, and $B := X \setminus (A_1 \cup A_2)$. Let

$$g(x) := \begin{cases} y_2 & \text{if } x \in A_1, \\ y_1 & \text{if } x \in A_2, \\ f(x) & \text{if } x \in B. \end{cases} \tag{6.50}$$

In other words, everything that f maps to y_1 is now mapped to y_2 and everything that f maps to y_2 is now mapped to y_1. Then $g : X \to Y$ also is surjective.

Proof:

We notice that A_1, A_2, B partition X into three mutually exclusive parts: $X = B \uplus A_1 \uplus A_2$

and that the sets $f(A_1) = \{y_1\}$, $f(A_2) = \{y_2\}$, $f(B) = Y \setminus \{y_1, y_2\}$

partition Y into $Y = f(B) \uplus f(A_1) \uplus f(A_2)$. (Do you see why $f(B) = Y \setminus \{y_1, y_2\}$?)

B and hence $f(B)$ might be empty but none of the other four sets are.

It follows that there is indeed a function value $g(x)$ for each $x \in X$ and there is exactly one such value, i.e., g in fact defines a mapping from X to Y.

The surjectivity of g follows from that of f and the fact that

$$Y = f(B) \cup f(A_1) \cup f(A_2) = g(B) \cup g(A_2) \cup g(A_1) \tag{6.51}$$

(see (6.19) on p. 105 in prop. 6.3 (Properties of the direct image)). ■

6.2 Indicator Functions

Sometimes it is advantageous to think of the subsets of a universal set Ω as "binary" functions $\Omega \to \{0,1\}$.

Definition 6.2 (indicator function for a set)**.** Let Ω be "the" universal set, i.e., we restrict our scope of interest to subsets of Ω. Let $A \subseteq \Omega$. Let $1_A : \Omega \to \{0,1\}$ be the function defined as

$$1_A(\omega) := \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A. \end{cases} \tag{6.52}$$

1_A is called the **indicator function** of the set A. [58] □

The above association of a subset A of Ω with its indicator function is a one–one correspondence:

Proposition 6.12. *Let $\mathscr{F}(\Omega,\{0,1\})$ denote the set of all functions $f:\Omega \to \{0,1\}$, i.e., all functions f with domain Ω for which the only possible function values $f(\omega)$ are zero or one.*

a. *The mapping*

$$F : 2^\Omega \to \mathscr{F}(\Omega,\{0,1\}), \quad \textit{defined as} \;\; F(A) := 1_A \tag{6.53}$$

which assigns to each subset of Ω its indicator function is injective.

b. *Let $f \in \mathscr{F}(\Omega,\{0,1\})$. Further, let $A := \{f = 1\} := f^{-1}(\{1\}) := \{a \in A : f(a) = 1\}$. Then $f = 1_A$.*

c. *The function F above is bijective and its inverse function is*

$$G : \mathscr{F}(\Omega,\{0,1\}) \to 2^\Omega, \quad \textit{defined as} \;\; G(f) := \{f = 1\}. \tag{6.54}$$

*Proof of **a**: This follows from **c** which will be proved below.*

*Proof of **b**: We have*

$$\begin{aligned} f(\omega) = 1 \;&\Leftrightarrow \omega \in \{f = 1\} \quad \textit{(def. of inverse image)} \\ &\Leftrightarrow \omega \in A \quad \textit{(because } A = \{f = 1\}\textit{)} \\ &\Leftrightarrow 1_A(\omega) = 1 \quad \textit{(def. of indicator function).} \end{aligned}$$

*It follows that $f(\omega) = 1$ if and only if $1_A(\omega) = 1$. As the only other possible function value is 0 we conclude that $f(\omega) = 0$ if and only if $1_A(\omega) = 0$. It follows that $f(\omega) = 1_A(\omega)$ for all $\omega \in \Omega$, i.e., $f = 1_A$. This proves **b**.*

*Proof of **c**: According to theorem 4.1 on p.83 about the characterization of inverse functions **c** is proved if we can demonstrate that F and G are inverse to each other. To prove this it suffices to show that*

$$G \circ F = id_{2^\Omega} \quad \textit{and} \quad F \circ G = id_{\mathscr{F}(\Omega,\{0,1\})}. \tag{6.55}$$

Let $A \subseteq \Omega$ and $A \in \Omega$. Then

$$G \circ F(A) = G(1_A) = \{1_A = 1\} = \{\omega \in \Omega : 1_A(\omega) = 1\} = \{\omega \in \Omega : \omega \in A\} = A.$$

[58] Some authors call this **characteristic function** of A and some choose to write χ_A or $\mathbb{1}_A$ instead of 1_A.

This proves $G \circ F = id_{2^\Omega}$. Let $\omega \in \Omega$. Then

$$\begin{aligned}\big(F \circ G(f)\big)(\omega) &= F(\{f=1\})(\omega) \;=\; 1_{\{f=1\}}(\omega) \\ &= \begin{cases}1 & \textit{iff}\ \omega \in \{f=1\}, \\ 0 & \textit{iff}\ \omega \notin \{f=1\}\end{cases} = \begin{cases}1 & \textit{iff}\ f(\omega)=1, \\ 0 & \textit{iff}\ f(\omega) \neq 1\end{cases} = \begin{cases}1 & \textit{iff}\ f(\omega)=1, \\ 0 & \textit{iff}\ f(\omega)=0\end{cases} = f(\omega).\end{aligned}$$

*The equation next to the last results from the fact that the only possible function values for f are 0 and 1. It follows that $F \circ G(f) = f = id_{\mathscr{F}(\Omega,\{0,1\})}(f)$ for all $f \in \mathscr{F}(\Omega,\{0,1\})$, hence $F \circ G = id_{\mathscr{F}(\Omega,\{0,1\})}$. We have proved (6.55) and hence **c**.* ■

The following definition is a special case of "modular arithmetic". We will only use it in the context of indicator functions of set differences. For further information we refer to [1] B/G, ch.6.3.

Definition 6.3 (mod 2 addition)**.** Let $m, n \in \mathbb{Z}$. We define

(6.56)
$$m+n \mod 2 \;:=\; \begin{cases}0 & \text{if } m+n \text{ is even,} \\ 1 & \text{if } m+n \text{ is odd,}\end{cases}$$

and we call $m+n \mod 2$ the **sum mod 2** of m and n. □

Proposition 6.13. *Let $m, n, p \in \mathbb{Z}$. Then addition mod 2 is associative, i.e.,*

(6.57)
$$\big(m+n \mod 2\big) + p \mod 2 \;=\; m + \big(n+p \mod 2\big) \mod 2.$$

The proof is left as exercise 6.6.

Proposition 6.14. *Let A, B, C be subsets of Ω. Then*

(6.58)
$$\mathbb{1}_{A\cup B} = \max(\mathbb{1}_A, \mathbb{1}_B),$$
(6.59)
$$\mathbb{1}_{A\cap B} = \min(\mathbb{1}_A, \mathbb{1}_B),$$
(6.60)
$$\mathbb{1}_{A^\complement} = 1 - \mathbb{1}_A,$$
(6.61)
$$\mathbb{1}_{A\triangle B} = \mathbb{1}_A + \mathbb{1}_B \mod 2.$$

Proof: The proof of the first three equations is left as an exercise.

Proof of (6.61): This follows easily from the the fact that

$$(A \triangle B)^\complement \;=\; \{\omega \in \Omega : \textit{ either } \omega \in A \cap B \textit{ or } \omega \in \textit{ neither } A \textit{ nor } B\} \;\blacksquare$$

Proposition 6.15 (Addition mod 2 is associative)**.** *Let $A, B, C \subseteq \Omega$. Then*

(6.62)
$$(A \triangle B) \triangle C = A \triangle (B \triangle C).$$

Proof: This follows easily from (6.61) and prop.6.13 as follows. Let $\omega \in \Omega$. Then

$$\omega \in (A\triangle B)\triangle C \;\Leftrightarrow \mathbb{1}_{(A\triangle B)\triangle C} = 1 \;\Leftrightarrow\; \mathbb{1}_{A\triangle(B\triangle C)} = 1 \;\Leftrightarrow\; \omega \in A\triangle(B\triangle C).$$

We obtained the equivalence in the middle from prop.6.13. ■

6.3 Exercises for Ch.6

Exercise 6.1. Prove prop.6.1 on p.104:

a. $f(\emptyset) = f^{-1}(\emptyset) = \emptyset$
b. $A_1 \subseteq A_2 \subseteq X \Rightarrow f(A_1) \subseteq f(A_2)$
c. $B_1 \subseteq B_2 \subseteq Y \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$
d. $x \in X \Rightarrow f(\{x\}) = \{f(x)\}$
e. $f(X) = Y \Leftrightarrow f$ is surjective
f. $f^{-1}(Y) = X$ always! □

Exercise 6.2. Prove (6.31) of prop.6.4 on p.106: Let X, Y, Z be an arbitrary, non–empty sets. Let $f : X \to Y$ and $g : Y \to Z$ and let $W \subseteq Z$. Then $(g \circ f)(U) = g\big(f(U)\big)$ for all $U \subseteq X$. □

Exercise 6.3. Prove prop.6.7 on p.108.
Hint: The main tools you need are prop.6.5 on p.107, prop6.6 on p.108, and their corollaries. □

Exercise 6.4. Prove prop.6.8 on p.108.
Hint: Work with the inverse of f and apply prop.6.2 on p.104. □

Exercise 6.5. Prove prop.6.9 on p.109.
Hint: To prove **a**, use prop.6.3 on p.105. □

Exercise 6.6. Prove prop.6.13 on p.112: Let $m, n, p \in \mathbb{Z}$. Then

$$\big(m+n \mod 2\big) + p \mod 2 \quad = \quad m + \big(n+p \mod 2\big) \mod 2.$$

Hint: There are eight possible combinations of zeroes and ones for the functions

$$(m,n,p) \to \big(m+n \mod 2\big) + p \mod 2 \quad \text{and} \quad (m,n,p) \to m + \big(n+p \mod 2\big) \mod 2.$$

Complete the entries in the table below and show that the entries in the two rightmost columns match. To save space, write $m \oplus n$ for $m+n \mod 2$. To get you started, the row for $m = 1, n = 0, p = 0$ has been already completed.

m	n	p	$m \oplus n$	$n \oplus p$	$(m \oplus n) \oplus p$	$m \oplus (n \oplus p$
0	0	0				
0	0	1				
0	1	0				
0	1	1				
1	0	0	1	0	1	1
1	0	1				
1	1	0				
1	1	1				

□

7 Some Miscellaneous Topics

Although this chapter only contains a single topic at this time (cardinality), additional topics are planned in the future.

7.1 Cardinality - Alternate Approach to Beck/Geoghegan (Study this!)

> ***Notation:*** *In this entire chapter on cardinality, if $n \in \mathbb{N}$, the symbol $[n]$ does not denote an equivalence class of any kind but the set $\{1, 2, \dots, n\}$ of the first n natural numbers.*

Definition 7.1 (cardinality comparisons)**.** Given are two arbitrary sets X and Y.

We say X, Y **have same cardinality** and we write $\textbf{card}(X) = \textbf{card}(Y)$

if there is a bijective mapping $f : X \xrightarrow{\sim} Y$.

We say **cardinality of** $X \leqq$ **cardinality of** Y and we write $\textbf{card}(X) \leqq \textbf{card}(Y)$ if there is an injective mapping $f : X \to Y$.

Finally we say **cardinality of** $X <$ **cardinality of** Y and we write $\textbf{card}(X) < \textbf{card}(Y)$ if both $\text{card}(X) \leqq \text{card}(Y)$ and $\text{card}(Y) \neq \text{card}(X)$, i.e., there is an injective mapping but not a bijection $f : X \to Y$. □

Remark 7.1. Note that the above definition does not specify how $\textbf{card}(X)$ itself is defined. This will be done in def.7.3 on p.121. □

Proposition 7.1.

> *Let X and Y be two non-empty sets and $f : X \xrightarrow{\sim} Y$. Then*
> ***a.*** *Y is finite if and only if X is finite,*
> ***b.*** *Y is countably infinite if and only if X is countably infinite,*
> ***c.*** *Y is countable if and only if X is countable,*
> ***d.*** *Y is uncountable if and only if X is uncountable.*

*Proof. The proof of **a** and **b** is based on prop.4.1.**c** on p.84 which states that the composition of two bijective functions is bijective.*

We only need to prove the " $\Rightarrow$" directions because we obtain " $\Leftarrow$" by switching the roles of X and Y.

*Proof of **a**. If X is finite then there exists $n \in \mathbb{N}$ and a bijection $g : X \xrightarrow{\sim} [n]$. $Y \xrightarrow{f^{-1} \circ g} [n]$ is bijective according to prop.4.1.**c** on p.84. This proves that Y is finite.*

*Proof of **b**. If X is countably infinite then there exists a bijection $g : X \xrightarrow{\sim} \mathbb{N}$. Because $Y \xrightarrow{f^{-1} \circ g} \mathbb{N}$ is bijective, Y also is countably infinite.*

*Proof of **c**. If X is countable then this set is either finite or countably infinite. If X is finite then Y is finite according to part **a**; if X is countably infinite then Y is countably infinite according to part **b**. This proves that Y is countable.*

*Proof of **d**. Assume to the contrary that X is uncountable and Y is countable. It follows from part **c** that X is countable and we have reached a contradiction. This proves that Y is uncountable.* ■

Proposition 7.2. *Let $m, n \in \mathbb{N}$. Let $\emptyset \neq A \subseteq [m]$. If $m < n$ then there is no surjection from A to $[n]$.*

Proof by induction on n:

Base case: Let $n = 2$. This implies $m = 1$ and $A = [1]$ (no other non-empty subset of $[1]$). For an arbitrary function $f : A \to [2]$ we have either $f(1) = 1$ in which case $2 \notin f(A)$ or $f(1) = 2$ in which case $1 \notin f(A)$. This proves the base case.

Induction assumption: Fix $n \in \mathbb{N}$ and assume that for any $\tilde{m} < n$ and non-empty $\tilde{A} \subseteq [\tilde{m}]$ there is no surjective $\tilde{f} : \tilde{A} \to [n]$.

We must prove the following: Let $m \in \mathbb{N}$ and $\emptyset \neq A \subseteq [m]$. If $m < n + 1$ then there is no surjection from A to $[n + 1]$.

We now assume to the contrary that a surjective $f : A \to [n + 1]$ exists.

case 1:** $n \notin A$**:

As $m < n + 1$ this implies both $n, n + 1 \notin A$, hence $A \subseteq [n - 1]$.

Let $\tilde{A} := A \setminus f^{-1}\{n + 1\}$. Then $\tilde{A} \subseteq A \subseteq [n - 1]$. Let $\tilde{f} := f\Big|_{\tilde{A}}$ be the restriction of f to $\tilde{A}$.

Because the surjective f "hits" every integer between 1 and $n + 1$ and we only removed those $a \in A$ which map to $n + 1$, $\tilde{f}$ covers any integer between 1 and n.

In other words, $\tilde{f} : \tilde{A} \to [n]$ is surjective, contradictory to our induction assumption.

***case 2:** $n \in A$ **and** $f(n) = n + 1$**:** As in case 1, let $\tilde{A} := A \setminus f^{-1}\{n + 1\}$.*

Then $\tilde{A} \subseteq [n - 1]$ because n was discarded from A as an element of $f^{-1}\{n + 1\}$.

Again, the surjective f maps to every integer between 1 and $n + 1$, and again, we only removed those $a \in A$ which map to $n + 1$.

Hence $\tilde{f} : \tilde{A} \to [n]$ is surjective, contradictory to our induction assumption.

***case 3:** $n \in A$ **and** $f(n) \neq n + 1$**:** According to lemma 7.2 on p.115 we can replace f by a surjective function g which maps n to $n + 1$.*

This function satisfies the conditions of case 2 above, for which it was already proved that no surjective mapping from A to $[n + 1]$ exists. We have reached a contradiction. ■

Corollary 7.1 (No bijection from $[m]$ to $[n]$ exists)**.** *B/G Thm.13.4: Let $m, n \in \mathbb{N}$. If $m \neq n$ then there is no bijective $f : [m] \xrightarrow{\sim} [n]$.*

Proof: We may assume $m < n$ and can now apply prop. 7.2 with $A := [m]$. ■

Corollary 7.2 (Pigeonhole Principle)**.** *B/G Prop.13.5: Let $m, n \in \mathbb{N}$. If $m < n$ then there is no injective $f : [n] \to [m]$.*

Proof: Otherwise, by thm.4.2 on p.85, f would have a (surjective) left inverse $g : [m] \to [n]$ in contradiction to the preceding proposition. ■

Proposition 7.3 (B/G Prop.13.6, p.122: Subsets of finite sets are finite)**.** *Let $\emptyset \neq B \subseteq A$ and let A be finite. Then B is finite.*

Proof: Done by induction on the number of elements n of A:

Base case: $n = 1$ or $n = 2$: Proof obvious.

Induction assumption: Assume that all subsets of sets of cardinality less than n are finite.

Now let A be a set with cardinality card$(A) = n$. there is a bijection $a(\cdot) : [n] \xrightarrow{\sim} A$. Let $B \subseteq A$.

***Case 1:** $a(n) \in B$**:** Let $B_n := B \setminus \{a(n)\}$ and $A_n := A \setminus \{a(n)\}$.*

Then the restriction $a(\cdot)\Big|_{[n-1]}$ of $a(\cdot)$ to $[n-1]$ is a bijection $[n-1] \xrightarrow{\sim} A_n$ according to cor.6.3 on p.109.

As card$(A_n) = n-1$ and $B_n \subseteq A_n$ it follows from the induction assumption that B_n is finite: there exists $m \in \mathbb{N}$ and a bijection $b(\cdot) : [m] \xrightarrow{\sim} B_n$.

We now extend $b(\cdot)$ to $[m+1]$ by defining $b(m+1) := a(n)$. It follows that this extension remains injective and it is also surjective if we choose as codomain $B_n \cup \{a(n)\} \; = B$.

It follows that B is finite.

***Case 2:** $a(n) \notin B$**:** We pick an arbitrary $b \in B$. Let $j := a^{-1}(b)$. Clearly $j \in [n]$.*

We modify the mapping $a(\cdot)$ by switching the function values for j and n. We obtain another bijection $f : [n] \xrightarrow{\sim} A$ (see prop.6.10 on p. 109) for which $f(n) = a(j) = b \in B$.

We apply to f what was proved in case 1 and obtain that B is finite. ■

Our next major result is that subsets of countable sets are countable.

Lemma 7.1. *Any subset of $\mathbb{N}$ is countable.*

Proof: Let $B \subseteq \mathbb{N}$. Finite sets are countable by definition, so we may assume that B is infinite. We extract $j_1 < j_3 < j_2 < \dots$ from $\mathbb{N}$ as follows.

$$\begin{aligned} j_1 &:= \min\{j \in \mathbb{N} : j \in B\}, \\ j_2 &:= \min\{j \in \mathbb{N} : j > j_1 \text{ and } : j \in B\}, \\ &\dots\dots\dots\dots\dots\dots\dots \\ j_{k+1} &:= \min\{j \in \mathbb{N} : j > j_k \text{ and } : j \in B\} \end{aligned}$$

Note that none of the sets $B_k := \{j \in \mathbb{N} : j > j_k \text{ and } j \in B\}$ are empty because $B_k = \emptyset$ would imply $B \subseteq \{1, 2, \dots, j_k\}$, i.e., B is a subset of the finite set $[j_k]$. This is a contradiction to prop.7.3 (Subsets of finite sets are finite) on p.116.

Let $B' := \{b_k : k \in \mathbb{N}\}$. Note that $j_k < j_{k+1}$ for all $k \in \mathbb{N}$ by construction. Hence

$$j_k \geqq k \quad \text{for all } k \in \mathbb{N}. \tag{7.1}$$

In particular, B' is not bounded above.

We claim that $B = B'$. It follows from $n_k \in B$ for all k that $B' \subseteq B$. We now show that $B \subseteq B'$. Otherwise there would be $n' \in B$ such that $n' \neq j_k$ for all natural numbers k. It follows from 7.1 that $n' \leqq j_{n'}$. Hence there is $m \leqq n'$ such that $j_{m-1} < n' < j_m$. This contradicts

$$j_m \;=\; \min\{j \in \mathbb{N} : j > j_{m-1} \text{ and } : j \in B\}.$$

It follows from 7.1 that the function

$$\psi : \mathbb{N} \to B; \qquad k \mapsto j_k$$

is injective. ψ also is surjective because its codomain is $B' = \{b_k : k \in \mathbb{N}\}$. Hence B is countably infinite, hence B is countable. ■

Theorem 7.1. *Subsets of countable sets are countable.*

Proof: Let X be countable and $A \subseteq X$. We may assume that X is infinite because otherwise A is finite as the subset of a finite set. See prop.7.3 on p.116. We also may assume that A is infinite because finite sets are countable by definition.

X is countably infinite, hence there exists a bijection

$$\phi : \mathbb{N} \xrightarrow{\sim} X; \qquad n \mapsto x_n := x(n).$$

Let $M := \phi^{-1}(A) = \{n \in \mathbb{N} : x_n \in A\}$. Let

$$\psi : A \to M; \quad x \mapsto \psi(x) := \phi^{-1}(x), \qquad \text{i.e.,} \;\; \psi \;=\; A \xrightarrow{\phi^{-1}\big|_A} \phi^{-1}(A).$$

ψ is injective as the restriction of the bijective, hence injective function ϕ^{-1}. ψ also is surjective because its codomain is $\phi^{-1}(A)$. It follows that ψ is bijective.

According to lemma 7.1, M is countable. Assume that M is finite. Then there exists $n \in \mathbb{N}$ and a bijection $F : M \xrightarrow{\sim} [n]$. The composition $F \circ \psi : A \xrightarrow{\sim} M \xrightarrow{\sim} [n]$ is bijective (prop.4.1 on p.84). It follows that A must be finite — a contradiction to the assumption that A is infinite.

We have proved that the countable set M is not finite, hence countably infinite. There exists a bijection $G : M \xrightarrow{\sim} \mathbb{N}$. We conclude as before that the composition $G \circ \psi : A \xrightarrow{\sim} M \xrightarrow{\sim} \mathbb{N}$ is bijective. It follows that A is countably infinite, hence countable. ■

The next proposition will hep us to prove that countable unions of countable sets are countable.

Proposition 7.4 (B/G Cor.13.16, p.122)**.** *$\mathbb{N}^2$ is countable.*

Proof: [59] *Done by directly specifiying a bijection $F : \mathbb{N}^2 \xrightarrow{\sim} \mathbb{N}$.*

The following definitions and observations will make it easier to understand this proof. Let

$$s_0 \;:= 0; \qquad s_n \;:=\; \sum_{j=1}^{n} j \;=\; 1 + 2 + \cdots + n \;=\; \frac{n(n+1)}{2}.$$

[59] Understanding this proof is not very important and you will understand the essence of it if you read instead the subsequent remark 7.2.

For the last equality see [1] B/G prop.4.11 on p.37. We note that

$$s_{n-1} + n = s_n. \tag{7.2}$$

Moreover,

$$A_n := \{j \in \mathbb{N} : s_{n-1} < j \leqq s_n\} = \{s_{n-1}+1, s_{n-1}+2, \dots, s_{n-1}+n\} \quad (n \in \mathbb{N}) \tag{7.3}$$

is a partition of $\mathbb{N}$ *(see example 5.2.**d** on p.97).*

For $n \in \mathbb{N} \setminus \{1\} = \{2, 3, 4, \dots\}$ *let*

$$D_n := \{(i, j) \in \mathbb{N}^2 : i + j = n\}$$

be the set of all pairs of natural numbers whose sum equals n*. Clearly,* $(D_{n+1})_{n \geqq 2}$ *is a partition of* $\mathbb{N}^2$*.*

For $n \in \mathbb{N}$ *let*

$$f_n : A_n \to D_{n+1}, \qquad s_{n-1} + k \mapsto f_n(s_{n-1} + k) := (n + 1 - k, k) \quad (k \in \mathbb{N}, 1 \leqq k \leqq n). \tag{7.4}$$

We see from the second equality in (7.3) that the set of argument values $s_{n-1} + k$ $(1 \leqq k \leqq n)$ *coincides with the domain* A_n *and it follows that (7.4) indeed defines a function* $A_n \to \mathbb{N}^2$*.*

It is immediate that $D_{n+1} = \{(n + 1 - k, k) : 1 \leqq k \leqq n\}$*. We conclude that* $f_n(A_n)) = D_{n+1}$ *and we have proved surjectivity of* f_n*.*

Finally we observe that if $1 \leqq k, k' \leqq n$ *and* $k \neq k'$ *then*

$$f_n(s_{n-1} + k) = (n + 1 - k, k) \neq (n + 1 - k', k') = f_n(s_{n-1} + k')$$

and this proves injectivity of f_n*.*

We now "glue together" the functions f_n *to obtain*

a function f *with domain* $\bigcup \Big[A_n : n \in \mathbb{N}\Big] = \mathbb{N}$ *and codomain* $\bigcup \Big[D_{n+1} : n \in \mathbb{N}\Big] = \mathbb{N}^2$ *as follows:*

$f(m) := f_n(m)$ *for* $m \in A_n$*, i.e.,*

$$f(s_{n-1} + k) = f_n(s_{n-1} + k) = (n + 1 - k, k) \text{ for } k \in A_n.$$

f *inherits injectivity from the individual* f_n *as the ranges* $f_n(A_n) = D_{n+1}$ *are mutually disjoint for different values of* n *and* f *inherits surjectivity from the* f_n *as*

$$f(\mathbb{N}) = \bigcup \Big[f_n(A_n) : n \in \mathbb{N}\Big] = \mathbb{N}^2 = \bigcup \Big[f(A_n) : n \in \mathbb{N}\Big] = \mathbb{N}^2$$

To summarize, we have proved that f *is a bijective mapping between* $\mathbb{N}$ *and* $\mathbb{N}^2$ *and this proves that* $\mathbb{N}^2$ *is countable.* ■

Remark 7.2. The following will help to visualize the proof just given. We think of $\mathbb{N}^2$ as a matrix with "infinitely many rows and columns"

$$(1,1)\ (1,2)\ (1,3)\ \dots \tag{7.5}$$

$$(2,1)\ (2,2)\ (2,3)\ \dots \tag{7.6}$$

$$(3,1)\ (3,2)\ (3,3)\ \dots \tag{7.7}$$

We reorganize this matrix into an ordinary sequence $(f(j))_{j\in\mathbb{N}}$ as follows:

(7.8) $f(1) = f_1(1) = (1,1),$

(7.9) $f(2) = f_2(2) = (1,2),\ f(3) = f_2(3) = (2,1),$

(7.10) $f(4) = f_3(4) = (1,3),\ f(5) = f_3(5) = (2,2),\ f(6) = f_3(6) = (3,1),$

(7.11) $f(7) = f_4(7) = (1,4),\ f(8) = f_4(8) = (2,3),\ f(9) = f_4(9) = (3,2), f(10) = f_4(10) = (4,1),$

(7.12) $\dots$

In other words, we traverse first D_2, then D_3, then D_4, ... starting for each D_n at the upper right $(1, n-1)$ and ending at the lower left $(n-1, 1)$). $\square$

Theorem 7.2 (B/G prop.13.6: Countable unions of countable sets)**.** *The union of countably many countable sets is countable.*

Proof: Let the sets

$$A_1,\ A_2,\ A_3,\ \dots\ \text{be countable and let}\ A := \bigcup_{n\in\mathbb{N}} A_i.$$

We assume that at least one of those sets is not empty: otherwise their union is empty, hence finite, hence countable and we are done.

We may assume, on account of prop.5.3 that the sets are mutually disjoint, i.e., any two different sets A_i, A_j have intersection $A_i \cap A_j = \emptyset$ (see definition (2.4) on p.12).

***A.** As each of the non-empty A_i is countable, either A is finite and we have an $N_i \in \mathbb{N}$ and a bijective mapping $a_i(\cdot) : A_i \xrightarrow{\sim} [N_i]$, or A_i is countably infinite and we have a bijective mapping $a_i(\cdot) : A_i \xrightarrow{\sim} \mathbb{N}$.*

We now define the mapping $f : A \to \mathbb{N}^2$ as follows: Let $a \in A$. As the A_j are disjoint there is a unique index i such that $a \in A_i$ and, as sets do not contain duplicates of their elements, there is a unique index j such that $a = a_i(j)$.

In other words, for any $a \in A$ there exists a unique pair $(i_a, j_a) \in \mathbb{N}^2$ such that $a = a_{i_a}(j_a)$ and the assignment $a \mapsto (i_a, j_a)$ defines an injective function $f : A \to \mathbb{N}^2$.

But then this same assignment gives us a bijective function $F : A \xrightarrow{\sim} f(A)$.

$f(A)$ is countable as a subset of the countable set $\mathbb{N}^2$ and this proves the theorem as any subset of a countable set is countable (see B/G prop.13.10). $\blacksquare$

Corollary 7.3. *Let the set X be uncountable and let $A \subseteq X$ be countable. Then the complement $A^{\complement}$ of A is uncountable.*

The proof is left as exercise **??***.*

Here is another corollary to thm.7.2.

Corollary 7.4. *The set Z of all integers is countable.*

Proof: The set $-\mathbb{N}$ is countable because the function $n \mapsto -n$ is a bijection $\mathbb{N} \xrightarrow{\sim} -\mathbb{N}$, hence

$$Z = \mathbb{N} \cup -\mathbb{N} \cup \{0\}$$

is countable as the union of three countable sets. ■

We now examine the cardinality of cartesian products.

Theorem 7.3 (Finite Cartesians of countable sets are countable)**.** *The Cartesian product of finitely many countable sets is countable.*

Proof by induction: Let $X := X_1 \times \cdots \times X_n$ *We may assume that none of the factor sets* X_j *is empty: Otherwise the Cartesian is empty too and there is nothing to prove.*

The proof is a triviality for $k = 1$*. It is more instructive to choose* $k = 2$ *for the base case instead.*

So let X_1, X_2 *be two nonempty countable sets. We now prove that* $X_1 \times X_2$ *is countable.*

For fixed $x_1 \in X_1$ *the function* $F_2 : X_2 \to \{x_1\} \times X_2; \quad x_2 \mapsto (x_1, x_2)$ *is bijective because it has as an inverse the function* $G_2 : \{x_1\} \times X_2 \to X_2; \quad (x_1, x_2) \mapsto x_2$*. It follows that* $\{x_1\} \times X_2$ *is countable.*

Hence $X_1 \times X_2 = \bigcup_{x \in X_1} \{x_1\} \times X_2$ *is countable according to thm.7.2 on p.119. We have proved the base case.*

Our induction assumption is that $X_1 \times \cdots \times X_k$ *is countable. We must prove that* $X_1 \times \cdots \times X_{k+1}$ *is countable. We can "identify"*

$$X_1 \times \cdots \times X_{k+1} = (X_1 \times \cdots \times X_k) \times X_{k+1} \tag{7.13}$$

by means of the bijection $(x_1, \dots, x_n, x_{n+1}) \mapsto \big((x_1, \dots, x_n), x_{n+1}\big)$*. According to the induction assumption the set* $X_1 \times \cdots \times X_k$ *is countable.*

The proof for the base case shows that $X_1 \times \cdots \times X_{k+1}$ *as the Cartesian product of the two sets* $X_1 \times \cdots \times X_k$ *and* X_{k+1} *is countable. This finishes the proof of the induction step.* ■

Corollary 7.5. *Let* $n \in \mathbb{N}$*. The sets* $\mathbb{Q}^n$ *and* $\mathbb{Z}^n$ *are countable.*

Proof: This follows from the preceding theorem because the sets $\mathbb{Q}$ *and* $\mathbb{Z}$ *are countable.* ■

Definition 7.2 (algebraic numbers)**.** Let $x \in \mathbb{R}$ be the root (zero) of a polynomial with integer coefficients. We call such x an **algebraic number** and we call any real number that is not algebraic a **transcendental number**. □

Proposition 7.5 (B/G Prop.13.21, p.125)**.** *The set of all algebraic numbers is countable.*

Proof: Let P *be the set of all integer polynomials and* Z *the set of zeroes for all such polynomials. Let*

$$P_n := \{\text{polynomials } p(x) = \sum_{j=0}^{n} a_j x^j : a_j \in \mathbb{Z} \text{ and } -n \leqq a_j \leqq n\}. \tag{7.14}$$

Then P_n *is finite and*

$$Z_n := \{x \in \mathbb{R} : p(x) = 0 \text{ for some } p \in P_n\} \tag{7.15}$$

also is finite as a polynomial of degree n *has at most* n *zeroes.*

Z is the countable union of the sets Z_n. It follows that Z is countable. ■

Here are some trivial consequences of the fact that $\mathbb{R}$ is uncountable (see thm. **??***, p.***??** *and B/G Thm.13.22).*

Proposition 7.6. *All transcendental numbers are uncountable.*

Proof: the uncountable real numbers are the disjoint union of the countable algebraic numbers with the transcendentals. The assertion follows from cor.7.3. ■

7.1.1 Cardinality as a Partial Ordering

We assume in this subchapter that all sets are subsets of a universal set Ω. Having such a universal set allows us to declare on its power set 2^Ω equivalence relations. If we work with specific sets, e.g. the set $\mathbb{R}$ of all real numbers, we assume implicitly that those sets are contained in Ω.

We defined in definition 7.1 on p114) the meaning of $\textbf{card}(X) = \textbf{card}(Y)$ and $\textbf{card}(X) \leqq \textbf{card}(Y)$ for two sets X and Y but we never defined the expression $\textbf{card}(X)$ per se. This will be done now.

Definition 7.3 (Cardinality as an equivalence class)**.** Let $X, Y \subseteq \Omega$. We say that X and Y are equivalent and we write $X \sim Y$ if and only if $X = Y$ or both X and Y are nonempty and there is a bijective function $f : X \to Y$ [60] The proposition following this definition shows that "$\sim$" is indeed an equivalence relation.

We now DEFINE for a set $X \subseteq \Omega$ its **cardinality** as follows:

$$\textbf{card}(X) := \{Y \subseteq \Omega : \exists \text{ bijection } X \to Y\}. \tag{7.16}$$

In other words, $\textbf{card}(X)$ is the equivalence class $[X]$ of X for the relation "$\sim$". □

Proposition 7.7. *$X \sim Y$ as defined above is an equivalence relation on 2^Ω.*

Proof: Left as an exercise.

■

The following theorem allows us to prove that the relation

$$\textbf{card}(X) \preceq \textbf{card}(Y) \Leftrightarrow \textbf{card}(X) \leqq \textbf{card}(Y),$$

i.e. there exists an injection $X \to Y$, is antisymmetric.

Theorem 7.4 (Cantor-Schröder-Bernstein's Theorem)**.** *Let $X, Y \subseteq \Omega$. Let there be functions $f' : X \to Y$ and $g' : Y \to X$ which both are injective. Then there exists a bijection $X \to Y$.*

Proof: The proof given here follows closely that of [1] B/G (Beck/Geoghegan), Further Topics F: Cardinal Number and Ordinal Number.

[60] We had to list the case $X = Y$ separately to capture the case $X = Y = \emptyset$ because functions, hence bijections, are not defined for empty domains or codomains.

We have no interest concerning any particulars of the sets X and Y. We only are interested in establishing the existence of a bijection $h : X \to Y$. We hence may assume that X and Y are mutually disjoint (see remark 4.12 on p.86).

Let $f : X \xrightarrow{\sim} f'(X)$ and $g : Y \xrightarrow{\sim} g'(Y)$ be the bijective functions obtained from the injections f' and g' by restricting their codomains to the images of their domains. We note that the subsets $f'(X) \subseteq Y$ and $g'(Y) \subseteq X$ also are disjoint and that $f(X) = f'(X),\ g(Y) = g'(Y)$. Let

$$\sigma : f(X) \biguplus g(Y) \to X \biguplus Y; \qquad z \mapsto \begin{cases} f^{-1}(z) & \text{if } z \in f(X), \\ g^{-1}(z) & \text{if } z \in g(Y), \end{cases}$$

i.e., $\sigma\big|_{f(A)} = f^{-1}$ and $\sigma\big|_{g(B)} = g^{-1}$. Note that if $y \in f(X)$ then $\sigma(y) \in X$; if $x \in g(Y)$ then $\sigma(x) \in Y$. We can create iterates

$$\sigma^2(z) = \sigma\big(\sigma(z)\big),\ \ \sigma^3(z) = \sigma\big(\sigma^2(z)\big),\ \ \dots, \sigma^{n+1}(z) = \sigma\big(\sigma^n(z)\big),\ \ \dots,$$

just as long as $\sigma^n(z) \in f(X) \biguplus g(Y)$. We further define σ^0 for all $z \in f(X) \biguplus g(Y)$ as $\sigma^0(z) := z$. We associate with each $z \in X \biguplus Y$ a "score" $N(z) \in \mathbb{Z}_{\geqq 0} \cup \{\infty\}$ as follows.

- ***a.*** *If $\sigma^k(z) \in f(X) \biguplus g(Y)$ for all $k \in \mathbb{N}$ then $N(z) := \infty$.*
- ***b.*** *If $\sigma^k(z) \notin f(X) \biguplus g(Y)$ for some $k \in \mathbb{N}$ then $N(z) := \min\{\, j \geqq 0 : \sigma^j(z) \notin f(X) \biguplus g(Y)\,\}$.*

*Note that **b** implies the following: If $z = \sigma^0(z) \notin f(X) \biguplus g(Y)$ then $N(z) = 0$.*

Depending on whether we start out with $x \in X$ or $y \in Y$, we obtain the following finite or infinite sequences:

$$\begin{aligned} &\text{if } x \in X : \quad x \xrightarrow{\sigma} f^{-1}(x) \xrightarrow{\sigma} g^{-1}(f^{-1}(x)) \xrightarrow{\sigma} f^{-1}(g^{-1}(f^{-1}(x))) \xrightarrow{\sigma} \dots, \\ &\text{if } y \in Y : \quad y \xrightarrow{\sigma} g^{-1}(y) \xrightarrow{\sigma} f^{-1}(g^{-1}(y)) \xrightarrow{\sigma} g^{-1}(f^{-1}(g^{-1}(y))) \xrightarrow{\sigma} \dots. \end{aligned}$$

If $N(z) < \infty$ then the sequence will terminate after $N(z)$ iterations. Let

$$\begin{aligned} X_E &:= \{x \in X : N(x) \text{ is even }\}, X_O := \{x \in X : N(x) \text{ is odd }\}, X_\infty := \{x \in X : N(x) = \infty\}, \\ Y_E &:= \{y \in Y : N(y) \text{ is even }\}, Y_O := \{y \in Y : N(y) \text{ is odd }\}, Y_\infty := \{y \in Y : N(y) = \infty\}. \end{aligned}$$

The above defines partitions $X = X_E \biguplus X_O \biguplus X_\infty$ and $Y = Y_E \biguplus Y_O \biguplus Y_\infty$ of X and Y.

Each of the functions f, g, σ changes the score of its argument from odd to even and from even to odd. Hence

$$\begin{aligned} &f(X_E) \subseteq Y_O,\ f^{-1}(Y_O) \subseteq X_E, \qquad f(X_O) \subseteq Y_E,\ f^{-1}(Y_E) \subseteq X_O, \\ &g(Y_E) \subseteq X_O,\ g^{-1}(X_O) \subseteq Y_E, \qquad g(Y_O) \subseteq X_E,\ g^{-1}(X_E) \subseteq Y_O, \\ &f(X_\infty) \subseteq Y_\infty,\ f^{-1}(Y_\infty) \subseteq X_\infty, \qquad g(Y_\infty) \subseteq X_\infty,\ g^{-1}(X_\infty) \subseteq Y_\infty, \end{aligned} \tag{7.17}$$

We define a bijection $h : X \xrightarrow{\sim} Y$ as follows:

$$h : X \to Y; \qquad x \mapsto \begin{cases} \sigma(x) = g^{-1}(x) & \text{if } x \in X_O \biguplus X_\infty, \\ f(x) & \text{if } x \in X_E. \end{cases}$$

Note that $g^{-1}(x)$ is defined for all $x \in X_O \biguplus X_\infty$ because we then have $N(x) > 0$.

We show that h is injective: Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$. There are three cases.
Case 1: Both $x_1, x_2 \in X_O \biguplus X_\infty$. Then $h(x_1) = g^{-1}(x_1) \neq g^{-1}(x_2) = h(x_2)$ because the bijectivity of g implies that of g^{-1}. In particular, g^{-1} is injective.

Case 2: Both $x_1, x_2 \in X_E$. Then $h(x_1) = f(x_1) \neq f(x_2) = h(x_2)$ because f is injective.

Case 3: $x_1 \in X_O \biguplus X_\infty$ and $x_2 \in X_E$. It follows from (7.17) that $h(x_1) = g^{-1}(x_1) \in Y_E \uplus Y_\infty$ and that $h(x_2) = f(x_2) \in Y_O$. Because $Y_E \uplus Y_\infty$ and Y_O have no elements in common, it follows that $h(x_1) \neq h(x_2)$. We have proved that h is injective.

We finally show that h is surjective: Let $y \in Y$. There are two cases.
Case 1: $y \in Y_E \biguplus Y_\infty$. It follows from (7.17) that $g(y) \in X_O \biguplus X_\infty$, hence $h(g(y)) = \sigma(g(y)) = g^{-1}(g(y)) = y$. We have found an item in X which is mapped by h to y.

Case 2: $y \in Y_O$. It follows from (7.17) that $f^{-1}(y) \in X_E$, hence $h(f^{-1}(y)) = f(f^{-1}(y)) = y$. Again we have found an item in X which is mapped by h to y. We have proved that h is surjective. ■

Corollary 7.6. *The relation*

$$\textbf{\textit{card}}(X) \preceq \textbf{\textit{card}}(Y) \;\Leftrightarrow\; \textbf{\textit{card}}(X) \leqq \textbf{\textit{card}}(Y)$$

on 2^Ω is a partial ordering.

*Proof: Reflexivity is obvious, antisymmetry follows from Cantor-Schröder-Bernstein and transitivity follows from prop.4.1.**a**: The composition of two injective functions is injective.* ■

Theorem 7.5. *Let $X, Y \subseteq \Omega$. Then $\textbf{\textit{card}}(X) \leqq \textbf{\textit{card}}(Y)$ or $\textbf{\textit{card}}(Y) \leqq \textbf{\textit{card}}(X)$*

*Proof: The proof will be given in thm.**??**, p.**??**, of ch.**??** (Applications of Zorn's Lemma).* ■

7.2 Addenda to Ch.7 - EMPTY!

EMPTY

8 The Completeness of the Real Numbers System (Study this!)

8.1 Minima, Maxima, Infima and Suprema

Definition 8.1 (Upper and lower bounds, maxima and minima). [61] Let $A \subseteq \mathbb{R}$. Let $l, u \in \mathbb{R}$. We call l a **lower bound** of A if $l \leqq a$ for all $a \in A$. We call u an **upper bound** of A if $u \geqq a$ for all $a \in A$.

We call A **bounded above** if this set has an upper bound and we call A **bounded below** if A has a lower bound. We call A **bounded** if A is both bounded above and bounded below.

A **minimum** (min) of A is a lower bound l of A such that $l \in A$. A **maximum** (max) of A is an upper bound u of A such that $u \in A$. □

The next proposition will show that min and max are unique if they exist. This makes it possible to write $\min(A)$ *or* $\min A$ *for the minimum of* A *and* $\max(A)$ *or* $\max A$ *for the maximum of* A.

Proposition 8.1. *Let $A \subseteq \mathbb{R}$. If A has a maximum then it is unique. If A has a minimum then it is unique.*

Proof for maxima: Let u_1 and u_2 be two maxima of A: both are upper bounds of A and both belong to A. As u_1 is an upper bound, it follows that $a \leqq u_1$ for all $a \in A$. Hence $u_2 \leqq u_1$. As u_2 is an upper bound, it follows that $u_1 \leqq u_2$ and we have equality $u_1 = u_2$. The proof for minima is similar. ■

Definition 8.2. Let $A \subseteq \mathbb{R}$. We define

$$
\begin{aligned}
A_{lowb} &:= \{l \in \mathbb{R} : l \text{ is lower bound of } A\} \\
A_{uppb} &:= \{u \in \mathbb{R} : u \text{ is upper bound of } A\}.
\end{aligned}
\tag{8.1}
$$

□

Remark 8.1. Note that A is bounded above if and only if $A_{uppb} \neq \emptyset$ and bounded below if and only if $A_{lowb} \neq \emptyset$. □

Axiom 8.1. (see [1] B/G axiom 8.52, p.83).

> **Completeness axiom for $\mathbb{R}$:** Let $A \subseteq \mathbb{R}$. If its set of upper bounds A_{uppb} is not empty then A_{uppb} has a minimum.

The above has the status of an axiom due to the fact that the real numbers usually are given axiomatically as an "archimedian ordered field" which satisfies the completeness axiom just stated.

Remark 8.2. A_{lowb} and/or A_{uppb} may be empty. Examples are $A = \mathbb{R}$, $A = \mathbb{R}_{>0}$, $A = \mathbb{R}_{<0}$. □

Definition 8.3. Let $A \subseteq \mathbb{R}$. If A_{uppb} is not empty then $\min(A_{uppb})$ exists by axiom 8.1 and it is unique by prop. 8.1. We write $\sup(A)$ or l.u.b.(A) for $\min(A_{uppb})$ and call this element of $\mathbb{R}$ the **supremum** or **least upper bound** of A.

We will see in cor.8.1 that, if A_{lowb} is not empty, then $\max(A_{lowb})$ exists and is unique by prop. 8.1. We write $\inf(A)$ or g.l.b.(A) for $\max(A_{lowb})$ and call this number the **infimum** or **greatest lower bound** of A. □

[61] The definitions are given for $\mathbb{Z}$ in def.**??** on p.**??**.

Proposition 8.2 (Duality of upper and lower bounds, min and max, inf and sup)**.** *Let $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Then the following is true for $-x$ and $-A = \{-y : y \in A\}$:*

$$
\begin{aligned}
&-x \text{ is a lower bound of } A \;\Leftrightarrow\; x \text{ is an upper bound of -}A,\\
&-x \in A_{\mathcal{uppb}} \;\Leftrightarrow\; x \in (-A)_{\mathcal{lowb}},\\
&-x = \sup(A) \;\Leftrightarrow\; x = \inf(-A),\\
&-x = \max(A) \;\Leftrightarrow\; x = \min(-A).
\end{aligned} \tag{8.2}
$$

We switch the roles of x and $-x$ and obtain four more equations:

$$
\begin{aligned}
&-x \text{ is an upper bound of } A \;\Leftrightarrow\; x \text{ is a lower bound of -}A,\\
&-x \in A_{\mathcal{lowb}} \;\Leftrightarrow\; x \in (-A)_{\mathcal{uppb}},\\
&-x = \inf(A) \;\Leftrightarrow\; x = \sup(-A),\\
&-x = \min(A) \;\Leftrightarrow\; x = \max(-A).
\end{aligned} \tag{8.3}
$$

Proof: A simple consequence of

$$-x \le y \;\Leftrightarrow\; x \geqq -y \quad \textit{and} \quad -x \geqq y \;\Leftrightarrow\; x \leqq -y. \qquad \blacksquare$$

Corollary 8.1. *Let $A \subseteq \mathbb{R}$. If A has lower bounds then $\inf(A)$ exists.*

Proof: According to the duality proposition prop.8.2, if A has lower bounds then $(-A)$ has upper bounds. It follows from the completeness axiom that $\sup(-A)$ exists. We apply once more prop.8.2 to prove that $\inf(A)$ exists: $\inf(A) = \sup(-A)$.

*Here are some examples. We define **for all three** of them $f(x) := -x$ and $g(x) := x$.*

Example 8.1 (Example a: Maximum exists)**.** Let $X_1 := \{t \in \mathbb{R} : 0 \leqq t \leqq 1\}$.
For each $x \in X_1$ we have $|f(x) - g(x)| = g(x) - f(x) = 2x$ and the biggest possible such difference is $g(1) - f(1) = 2,$. So $\max(X_1)$ exists and equals $\max(X_1) = 2$. ☐

Example 8.2 (Example b: Supremum is finite)**.** Let $X_2 := \{t \in \mathbb{R} : 0 \leqq t < 1\}$, i.e., we now exclude the right end point 1 at which the maximum difference was attained. For each $x \in X$ we have

$$|f(x) - g(x)| = g(x) - f(x) = 2x$$

and the biggest possible such difference is certainly bigger than

$$g(0.9999999999) - f(0.9999999999) = 1.9999999998.$$

If you keep adding $5,000$ 9s to the right of the argument x, then you get the same amount of 9s inserted into the result $2x$, so $2x$ comes closer than anything you can imagine to the number 2, without actually being allowed to reach it.

The supremum is still considered in a case like this to be 2. This precisely is the difference in behavior between the supremum $s := \sup(A)$ and the **maximum** $m := \max(A)$ of a set $A \subseteq \mathbb{R}$ of real numbers: For the maximum there must actually be at least one element $a \in A$ so that $a = \max(A)$.

For the supremum it is sufficient that there is a sequence $a_1 \leqq a_2 \leqq \dots$ which approximates s from below in the sense that the difference $s - a_n$ "drops down to zero" as n approaches infinity. We will not be more exact than this because doing so would require us to delve into the concepts of convergence and contact points. ☐

Example 8.3 (Example c: Supremum is infinite)**.** Let $X_3 := \mathbb{R}_{\geq 0} = \{t \in \mathbb{R} : 0 \leqq t\}$. For each $x \in X_3$ we have again $|f(x) - g(x)| = g(x) - f(x) = 2x$. But there is no more limit to the right for the values of x. The difference $2x$ will exceed all bounds and that means that the only reasonable value for $\sup\{|f(x) - g(x)| : x \in X_3\}$ is $+\infty$.

As in case b above, the $\max$ does not exist because there is no $x_0 \in X_3$ such that $|f(x_0) - g(x_0)|$ attains the highest possible value among all $x \in X_3$.

You should understand that even though $\sup(A)$ as best approximation of the largest value of $A \subseteq \mathbb{R}$ is allowed to take the "value" $+\infty$ or $-\infty$ this cannot be allowed for $\max(A)$.

How so? The infinity values are not real numbers, but, by definition of the maximum, if $\alpha := \max(A)$ exists, then $\alpha \in A$. In particular, the max must be a real number. $\square$

That last example motivates the following definition.

Definition 8.4 (Supremum and Infimum of unbounded and empty sets)**.** If A is not bounded from above then we define

$$\sup A = \infty \tag{8.4}$$

If A is not bounded from below then we define

$$\inf A = -\infty \tag{8.5}$$

Finally we define

$$\sup \emptyset = -\infty, \qquad \inf \emptyset = +\infty. \ \square \tag{8.6}$$

Proposition 8.3. *Let* $A \subseteq B \subseteq \mathbb{R}$*.*

- ***a.*** *Then* $\inf(A) \geqq \inf(B)$ *and* $\sup(A) \leqq \sup(B)$*.*
- ***b.*** *If both A and B have a min then* $\min(A) \geqq \min(B)$*. If both A and B have a max then* $\max(A) \leqq \max(B)$*.*

Proof:
b *follows from* ***a*** *because if a set has a minimum then it equals its infimum and if a set has a maximum then it equals its supremum*

We now prove ***a*** *for suprema. The proof for infima is similar. First we note that if at least one of the sets is empty or not bounded above then the proof is trivial.* [62] *We may assume that both A and B are not empty and bounded above.*

It follows from $A \subseteq B$ *that any upper bound of B also is an upper bound of A.*

In particular $\sup(B) \in A_{uppb}$*, hence* $\sup(A) = \min(A_{uppb}) \leqq \sup(B)$*.* ■

[62] Note that if A is not bounded above then the same holds for the superset B and that if B is empty then A is empty.

Definition 8.5 (Translation and dilation of sets of real numbers)**.** Let $A \subseteq \mathbb{R}$ [63] and $\alpha, b \in \mathbb{R}$. We define

(8.7) $$\lambda A + b \ := \{\lambda a + b : a \in A\}.$$

In particular, for $\lambda = \pm 1$, we obtain

(8.8) $$A + b \ = \{a + b : a \in A\},$$
(8.9) $$-A \ = \{-a : a \in A\}. \ \square$$

Proposition 8.4 (The sup of a set is positively homogeneous)**.** *Let A be a non–empty subset of $\mathbb{R}$ and let $\lambda \in \mathbb{R}_{\geqq 0}$. Then* [64]

(8.10) $$\sup(\lambda A) \ = \ \lambda \sup(A).$$

Proof: (8.10) *holds for $\lambda = 0$ because*

$$\sup(0A) \ = \ \sup(\{0\}) \ = \ 0 \ = \ 0 \cdot \sup(A).$$

So we may assume that $\lambda > 0$. If $B \subseteq \mathbb{R}$, let $B_{upp\flat} \ := \ \{u \in \mathbb{R} : u$ is upper bound of $B\}$. Note that

(8.11) $$u \in A_{upp\flat} \ \Leftrightarrow u \geqq a \ \forall a \in A \ \Leftrightarrow \ \lambda u \geqq \lambda a \ \forall a \in A \ \Leftrightarrow \ \lambda u \in (\lambda A)_{upp\flat}.$$

It follows from $\sup(A) \in A_{upp\flat}$ that $\lambda \sup(A) \in (\lambda A)_{upp\flat}$, hence $\lambda \sup(A) \geqq \min\big((\lambda A)_{upp\flat}\big) = \sup(\lambda A)$.

It remains to show that $\lambda \sup(A) \leqq \sup(\lambda A)$.
We substitute $\frac{v}{\lambda}$ for u in (8.11) *and obtain $\frac{v}{\lambda} \in A_{upp\flat} \ \Leftrightarrow \ v \in (\lambda A)_{upp\flat}$.*

It follows from $\sup(\lambda A) \in (\lambda A)_{upp\flat}$ that $\frac{\sup(\lambda A)}{\lambda} \in A_{upp\flat}$, hence $\frac{\sup(\lambda A)}{\lambda} \geqq \min\big(A_{upp\flat}\big) = \sup(A)$. This proves $\lambda \sup(A) \leqq \sup(\lambda A)$. ■

Definition 8.6 (bounded functions)**.** Given is a nonempty set X. A real-valued function $f(\cdot)$ with domain X is called **bounded from above** if there exists a (possibly very large) number $\gamma_1 > 0$ such that

(8.12) $$f(x) \quad < \quad \gamma_1 \qquad \text{for all arguments } x.$$

It is called **bounded from below** if there exists a (possibly very large) number $\gamma_2 > 0$ such that

(8.13) $$f(x) \quad > \quad -\gamma_2 \qquad \text{for all arguments } x.$$

It is called a **bounded function** if it is both bounded from above and below. It is obvious that if you set $\gamma := \max(\gamma_1, \gamma_2)$ then bounded functions are exactly those that satisfy the inequality

(8.14) $$|f(x)| \quad < \quad \gamma \qquad \text{for all arguments } x. \ \square$$

[63] See also def.**??** in ch.**??**

[64] Recall that $\lambda A \ = \ \{\lambda a : a \in A\}$. See def.8.5 on p.127.

We note that f is bounded if and only if its range $f(X)$ is a bounded subset of $\mathbb{R}$. We further note that we have defined infimum and supremum for any kind of set: empty or not, bounded above or below or not. We use those definitions to define infimum and supremum for functions, sequences and indexed families.

Definition 8.7 (supremum and infimum of functions)**.** Let X be an arbitrary set, $A \subseteq X$ a subset of X, $f : X \to \mathbb{R}$ a real function on X. Look at the set $f(A) = \{f(x) : x \in A\}$, i.e., the image of A under $f(\cdot)$.

The **supremum of** $f(\cdot)$ **on** A is then defined as

$$\sup_A f \;:=\; \sup_{x \in A} f(x) \;:=\; \sup\,(f(A)) \tag{8.15}$$

The **infimum of** $f(\cdot)$ **on** A is then defined as

$$\inf_A f \;:=\; \inf_{x \in A} f(x) \;:=\; \inf\,(f(A)). \;\square \tag{8.16}$$

Definition 8.8 (supremum and infimum of families)**.** Let $(x_i)_{i \in I}$ be an indexed family of real numbers x_i.

The **supremum of** $(x_i)_{i \in I}$ is then defined as

$$\sup\,(x_i) \;:=\; \sup_i\,(x_i) \;:=\; \sup\,(x_i)_i \;:=\; \sup\,(x_i)_{i \in I} \;:=\; \sup_{i \in I} x_i \;:=\; \sup\,\{x_i : i \in I\} \tag{8.17}$$

The **infimum of** $(x_i)_{i \in I}$ is then defined as

$$\inf\,(x_i) \;:=\; \inf_i\,(x_i) \;:=\; \inf\,(x_i)_i \;:=\; \inf\,(x_i)_{i \in I} \;:=\; \inf_{i \in I} x_i \;:=\; \inf\,\{x_i : i \in I\}. \;\square \tag{8.18}$$

The definition above for families extends to sequences (the special case of $I = \{k \in \mathbb{Z} : k \geq k_0$ for some $k_0 \in \mathbb{Z}$).

Definition 8.9 (supremum and infimum of sequences)**.** Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of real numbers x_n. The **supremum of** $(x_n)_{n \in \mathbb{N}}$ is then defined as

$$\sup\,(x_n) \;:=\; \sup\,(x_n)_{n \in \mathbb{N}} \;:=\; \sup_{n \in \mathbb{N}} x_n \;=\; \sup\,\{x_n : n \in \mathbb{N}\} \tag{8.19}$$

The **infimum of** $(x_n)_{n \in \mathbb{N}}$ is then defined as

$$\inf\,(x_n) \;:=\; \inf\,(x_n)_{n \in \mathbb{N}} \;:=\; \inf_{n \in \mathbb{N}} x_n \;=\; \inf\,\{x_n : n \in \mathbb{N}\}. \;\square \tag{8.20}$$

We note that the "duality principle" for min and max, sup and inf is true in all cases above: You flip the sign of the items you examine and the sup/max of one becomes the inf/min of the other and vice versa.

Proposition 8.5. *X be a nonempty set and $\varphi, \psi : X \to \mathbb{R}$ be two real valued functions on X. Let $A \subseteq X$. Then*

$$\sup\{\varphi(x) + \psi(x) : x \in A\} \leqq \sup\{\varphi(y) : y \in A\} + \sup\{\psi(z) : z \in A\}, \tag{8.21}$$

$$\inf\{\varphi(x) + \psi(x) : x \in A\} \geqq \inf\{\varphi(y) : y \in A\} + \inf\{\psi(z) : z \in A\}. \tag{8.22}$$

Proof:
We only prove (8.21). The proof of (8.22) is similar. [65]

Let $U := \{\varphi(x) + \psi(x) : x \in A\}$, $V := \{\varphi(y) : y \in A\}$, $W := \{\psi(z) : z \in A\}$. Let $x \in A$.

Then $\sup(V)$ is an upper bound of $\varphi(x)$ and $\sup(W)$ is an upper bound of $\psi(x)$,

hence $\sup(V) + \sup(W) \geqq \varphi(x) + \psi(x)$.

This is true for all $x \in A$, hence $\sup(V) + \sup(W)$ is an upper bound of U.

It follows that $\sup(V) + \sup(W)$ dominates the least upper bound $\sup(U)$ of U and this proves (8.21). ■

8.2 Convergence and Continuity in $\mathbb{R}$

You are familiar with the concepts of convergent sequences and continuous functions whose domain and codomain both are sets of real numbers from calculus. We discuss them here in a more rigorous fashion. Convergence and continuity will be generalized in later chapters from $\mathbb{R}$ to so–called metric spaces.

Definition 8.10 (convergence of sequences of real numbers)**.** We say that a sequence (x_n) of real numbers **converges** [66] to $a \in \mathbb{R}$ for $n \to \infty$ if almost all of the x_n will come arbitrarily close to a in the following sense:

For any $\delta \in \mathbb{R}$ (no matter how small) there exists (possibly extremely large) $n_0 \in \mathbb{N}$ such that

$$|a - x_j| < \delta \quad \text{for all} \quad j \geqq n_0. \tag{8.23}$$

We write either of

$$a = \lim_{n\to\infty} x_n \qquad \text{or} \qquad x_n \to a \tag{8.24}$$

and we call a the **limit** of the sequence (x_n). □

> *There is an equivalent way of expressing convergence towards a: No matter how small a "neighborhood" $]a - \delta, a + \delta[$ of a you choose: at most finitely many of the x_n will be located outside that neighborhood.*

Example 8.4. Some simple examples for convergence:

[65] (8.22)can also be deduced from (8.21) and the fact that $\inf\{\varphi(u) : u \in A\} = -\sup\{-\varphi(v) : v \in A\}$.
[66] We will define convergence of a sequence of items more general than real numbers in ch.10.1.4 (see def.10.11 (convergence of sequences in metric spaces) on p.190).

a. Let $x_n := 1/n$ $(n \in \mathbb{N})$. Then $x_n \to 0$ as $n \to \infty$.
b. Let $\alpha \in \mathbb{R}$ and $z_n := \alpha^2\pi$ $(n \in \mathbb{N})$. Then the sequence $(z_n)_n$ has limit $\alpha^2\pi$.
c. More generally let $z_n := x_0$ for some $x_0 \in \mathbb{R}$ $(n \in \mathbb{N})$. Then $\lim\limits_{n\to\infty} z_n = x_0$. $\square$

*Proof of **a**: If $\delta > 0$, let $n_0 :=$ some integer larger than $1/\delta$. Such a number exists because the natural numbers are not bounded above. Then*

$$|x_n - 0| = 1/n < \delta.$$

*Proof of **b** and **c**: Left as an exercise.* $\blacksquare$

The following proposition shows that the limit behavior of a sequence is a property of its tail, i.e., it does not depend on the first finitely many indices.

Proposition 8.6. *Let $x_n, y_n \in \mathbb{R}$ be two sequences of real numbers. Assume there is $K \in \mathbb{N}$ such that $x_n = y_n$ for all $n \geqq K$. Let $L \in \mathbb{R}$. Then*

$$\lim_{n\to\infty} x_n = L \;\Leftrightarrow\; \lim_{n\to\infty} y_n = L, \qquad \lim_{n\to\infty} x_n = \pm\infty \;\Leftrightarrow\; \lim_{n\to\infty} y_n = \pm\infty.$$

Proof:

***Case 1**: $\lim\limits_{n\to\infty} x_n = L$.*
Let $\delta > 0$. Then there exists $N' \in \mathbb{N}$ such that

$$|x_j - L| < \delta \text{ for all } j \geqq N'. \tag{8.25}$$

Let $N := \max(K, N')$. It follows from $N \geqq K$, $N \geqq N'$ $x_n = y_n$ for all $n \geqq K$ and (8.25) that

$$|y_j - L| = |x_j - L| < \delta \text{ for all } j \geqq N'. \tag{8.26}$$

This proves that y_n converges to L.

***Case 2**: $\lim\limits_{n\to\infty} x_n = \infty$.*
Let $M \in \mathbb{R}$. It follows from $x_n \to \infty$ and def.8.11 that there exists $N' \in \mathbb{N}$ such that

$$|x_j| > M \text{ for all } j \geqq N'. \tag{8.27}$$

Let $N := \max(K, N')$. It follows from $N \geqq K$, $N \geqq N'$ $x_n = y_n$ for all $n \geqq K$ and (8.27) that

$$|y_j - L| = |x_j - L| > M \text{ for all } j \geqq N'. \tag{8.28}$$

This proves that $\lim_{n\to\infty} y_n = \infty$.

***Case 3**: $\lim\limits_{n\to\infty} x_n = -\infty$.*
This is true according to the already proven case 2, applied to the sequences $(-x_n)_n$ and $(-y_n)_n$. $\blacksquare$

Proposition 8.7. *[See B/G prop.10.16]*

Let $(x_n)_n$ be a sequence of real numbers such that $\lim_{n\to\infty} x_n$ exists. Let $K \in \mathbb{N}$. For $n \in \mathbb{N}$ let $y_n := x_{n+K}$. Then $(y_n)_n$ has the same limit.

Proof: ***Case 1****:* $\lim_{n\to\infty} x_n = L \in \mathbb{R}$.

The proof is left as exercise 8.3.

Case 2*:* $\lim_{n\to\infty} x_n = \infty$.
Let $M \in \mathbb{R}$. *It follows from* $x_n \to \infty$ *and def.8.11 that there exists* $N \in \mathbb{N}$ *such that*

$$|x_j| > M \text{ for all } j \geqq N. \tag{8.29}$$

Let $j \geqq N$. *Then* $j + K \geqq N$ *and it follows from (8.29) that*

$$|y_j| = |x_{j+K}| > M. \tag{8.30}$$

This proves that $\lim_{n\to\infty} y_n = \infty$.

Case 3*:* $\lim_{n\to\infty} x_n = -\infty$.
This is true according to the already proven case 2, applied to the sequences $(-x_n)_n$ *and* $(-y_n)_n$. ■

Proposition 8.8 (convergent ⇒ bounded)**.** *Let* $(x_n)_n$ *be a sequence in* $\mathbb{R}$ *with limit* x. *Then this sequence is bounded.*

Proof: We must prove that there exists $K \in \mathbb{R}$ *such that* $|x_j| \leqq K$ *for all* $j \in \mathbb{N}$.

Let $\delta = 1$ *in (8.23) and it follows that there is* $n_0 \in \mathbb{N}$ *such that* $|x_j - x| < 1$ *for all* $j \geqq n_0$. *But then*

$$|x_j| = |(x_j - x) + x| \leqq |x_j - x| + |x| \leqq |x| + 1 \text{ for all } j \geqq n_0.$$

Let $K := \max(1, |x_1|, |x_2|, \dots, |x_{n_0-1}|$ *It follows that* $|x_j| \leqq K + x$ *for all* $j \in \mathbb{N}$. ■

The following proposition states that the product of a sequence which converges to zero and a bounded sequence converges to zero.

Proposition 8.9 (bounded · zero–convergent is zero–convergent)**.** *Let* $(x_n)_n$ *and* $(\alpha_n)_n$ *be two sequences in* $\mathbb{R}$ *and let* $\alpha \in \mathbb{R}$.
If $\lim_{n\to\infty} x_n = 0$ *and if* $|\alpha_j| \leqq \alpha$ *for all* $j \in \mathbb{N}$ *then*

$$\lim_{j\to\infty} (\alpha_j x_j) = 0. \tag{8.31}$$

Proof:

Case 1: $\alpha = 0$. *Then* $\alpha_j = 0$ *for all* $j \in \mathbb{N}$, *hence* $\alpha_j x_j = 0$. *For any* $\delta > 0$ *let* $n_0 = 1$. *Then*

$$|\alpha_j x_j - 0| = |\alpha_j x_j| = 0 < \delta.$$

This proves convergence $\alpha_j x_j \to 0$.

Case 2: $\alpha \neq 0$, *i.e.,* $|\alpha| > 0$. *Let* $\delta > 0$. *We must show that*

$$\text{there is } n_0 \in \mathbb{N} \text{ such that } |\alpha_j x_j| < \delta \text{ for all } j \in \mathbb{N} \text{ such that } j \geqq n_0. \tag{8.32}$$

Let $\varepsilon := \delta/|\alpha|$. Then $\varepsilon > 0$ and it follows from $\lim_{j\to\infty} x_j = 0$ that

(8.33) *there is $N \in \mathbb{N}$ such that $|x_j| < \varepsilon$ for all $j \in \mathbb{N}$ such that $j \geqq N$.*

But then $|\alpha_j\, x_j| = |\alpha_j| \cdot |x_j| < |\alpha| \cdot \varepsilon = \delta$ for all $j \in \mathbb{N}$ such that $j \geqq N$. We choose $n_0 := N$ and (8.32) follows. ■

It is very rare that you need to apply def.8.10 on p.129 to compute a limit. Rather, the previous proposition and the following set of rules are employed.

Proposition 8.10 (Rules of arithmetic for limits)**.** *Let $(x_n)_n$ and $(y_n)_n$ be two sequences in $\mathbb{R}$ and let $x, y, \alpha \in \mathbb{R}$.*
Let $\lim\limits_{j\to\infty} x_j = x$ and $\lim\limits_{j\to\infty} y_j = y$. Then

- ***a.*** $\lim\limits_{j\to\infty} \alpha = \alpha$,
- ***b.*** $\lim\limits_{j\to\infty} (\alpha \cdot x_j) = \alpha \cdot x$,
- ***c.*** $\lim\limits_{j\to\infty} (x_j + y_j) = x + y$,
- ***d.*** $\lim\limits_{j\to\infty} (x_j \cdot y_j) = x \cdot y$,
- ***e.*** *if $x \neq 0$ then* $\lim\limits_{j\to\infty} \dfrac{1}{x_j} = \dfrac{1}{x}$.

*Proof of **a**: Exercise 8.1.*

*Proof of **b**:*
Case 1: $\alpha = 0$. Then $\alpha\, x_j$ is the constant sequence $0, 0, \dots$ which converges to $0 = \alpha\, x$ — Done.

Case 2: $\alpha \neq 0$. Let $\delta > 0$. We must show that

(8.34) *there is $n_0 \in \mathbb{N}$ such that $|\alpha\, x_j - \alpha\, x| < \delta$ for all $j \in \mathbb{N}$ such that $j \geqq n_0$.*

Let $\varepsilon := \delta/|\alpha|$. Then $\varepsilon > 0$ and it follows from $\lim_{j\to\infty} x_j = x$ that

(8.35) *there is $N \in \mathbb{N}$ such that $|x_j - x| < \varepsilon$ for all $j \in \mathbb{N}$ such that $j \geqq N$.*

But then $|\alpha\, x_j - \alpha\, x| = |\alpha| \cdot |x_j - x| < |\alpha| \cdot \varepsilon = \delta$ for all $j \in \mathbb{N}$ such that $j \geqq N$. We choose $n_0 := N$ and (8.34) is proved.

*Proof of **c**:*
Let $\delta > 0$. It follows from $\lim\limits_{j\to\infty} x_j = x$ and $\lim\limits_{j\to\infty} y_j = y$ that there exist $N_1, N_2 \in \mathbb{N}$ such that

(8.36) *if $j \geqq N_1$ then $|x_j - x| < \delta/2$ and if $j \geqq N_2$ then $|y_j - y| < \delta/2$.*

It follows from the triangle inequality $|A + B| \leqq |A| + |B|$ (prop.2.3 on p.17) and from (8.36) that

(8.37) $$|(x_j + y_j) - (x + y)| = |(x_j - x) + (y_j + y)| \leqq |x_j - x| + |y_j - y)| < \delta/2 + \delta/2 = \delta$$

for all $j \geqq \max(N_1, N_2)$. *Let* $n_0 := \max(N_1, N_2)$. *It follows from* (8.37) *that* $|(x_j + y_j) - (x + y)| < \delta$ *for all* $j \geqq n_0$. *This proves* ***c***.

Proof of ***d***:
Let $u_j := (x_j - x)y_j$ *and* $v_j := x(y_j - y)$ $(j \in \mathbb{N})$. *Then*

$$x_j y_j - xy \;=\; (x_j y_j - x y_j) + (x y_j - xy) \;=\; (x_j - x)y_j \;+\; x(y_j - y) \;=\; u_j - v_j. \tag{8.38}$$

The convergent sequences $(y_n)_n$ *and* $(x)_n$ *(constant sequence!) are bounded by prop.*8.8 *and it follows from parts* ***c*** *and* ***a*** *that*

$$\lim_{j\to\infty}(x_j - x) \;=\; \lim_{j\to\infty} x_j \;-\; \lim_{j\to\infty} x \;=\; x - x \;=\; 0,$$
$$\lim_{j\to\infty}(y_j - y) \;=\; \lim_{j\to\infty} y_j \;-\; \lim_{j\to\infty} y \;=\; y - y \;=\; 0,$$

i.e., the sequences $x_j - x$ *and* $y_j - y$ *converge to zero.*

*It now follows from prop.*8.9 *that* $\lim\limits_{j\to\infty} u_j \;=\; 0$ *and* $\lim\limits_{j\to\infty} v_j \;=\; 0$. *According to* (8.38) $x_j y_j \;=\; xy + u_j + v_j$ *is the limit of three convergent sequences.* [67] *It follows from part* ***c*** *that*

$$\lim_{j\to\infty} x_n y_n \;=\; \lim_{j\to\infty}(xy) \;+\; \lim_{j\to\infty} u_j \;+\; \lim_{j\to\infty} v_j \;=\; xy + 0 + 0 \;=\; xy.$$

Proof of ***e***:
Because $\lim\limits_{j\to\infty} x_n = x$ *and* $|x| > 0$ *there exists* $N_1 \in \mathbb{N}$ *such that* $|x_n - x| \leqq |x|/2$ *for all* $j \geqq N_1$. *Hence*

$$\begin{gathered} |x| = |(x - x_n) + x_n| \;\leqq\; |x - x_n| + |x_n| \;\leqq\; \frac{|x|}{2} + |x_n| \\ \Rightarrow \frac{|x|}{2} \;\leqq\; |x_n| \;\Rightarrow\; |x|\,|x_n| \;\geqq\; \frac{x^2}{2} \;\Rightarrow\; \frac{1}{|x x_n|} \;\leqq\; \frac{2}{x^2}. \end{gathered} \tag{8.39}$$

Let $z_n \;:=\; (x\,x_n)^{-1}$ *and* $K \;:=\; \max(2/x^2, |z_1|, |z_2|, \dots, |z_{N_1}|)$. *It follows from* (8.39) *that the sequence* $(z_n)_n$ *is bounded by* K. *It follows from part* ***a*** *that* $\lim\limits_{j\to\infty} x \;=\; x$ *and from part* ***c*** *that* $\lim\limits_{j\to\infty}(x_n - x) \;=\; \lim\limits_{j\to\infty}(x_n) - x \;=\; 0$.

*It now follows from prop.*8.9 *that*

$$\lim_{j\to\infty}\left(\frac{1}{x_n} - \frac{1}{x}\right) \;=\; \lim_{j\to\infty} \frac{1}{x\,x_n} \cdot (x \;-\; x_n) \;=\; \lim_{j\to\infty} z_n\,(x - x_n) \;=\; 0. \tag{8.40}$$

Let $\delta > 0$. *On account of* (8.40) *there exists* $n_0 \in \mathbb{N}$ *such that*

$$\left|\frac{1}{x_n} - \frac{1}{x}\right| \;=\; \left|\left(\frac{1}{x_n} - \frac{1}{x}\right) - 0\right| \;<\; \delta \;\text{ for all } j \geqq n_0. \tag{8.41}$$

This proves convergence of $1/x_n$ *to* $1/x$. ■

[67] The constant sequence (xy) has limit xy according to part **a**

Exercise 8.1. Prove example 8.4 part **c**: Let $z_n := x_0$ for some $x_0 \in \mathbb{R}$ ($n \in \mathbb{N}$). Then $\lim\limits_{n\to\infty} z_n = x_0$.

If that is too abstract, try to prove the special case **b** first. □

Proposition 8.11.

a. *Let x_n be a sequence of real numbers that is non-decreasing, i.e., $x_n \leqq x_{n+1}$ for all n (see def.* **??** *on p.***??** *), and bounded above. Then $\lim\limits_{n\to\infty} x_n$ exists and coincides with $\sup\{x_n : n \in \mathbb{N}\}$*

b. *Further, if y_n is a sequence of real numbers that is non-increasing, i.e., $y_n \geqq y_{n+1}$ for all n, and bounded below, the analogous result is that $\lim\limits_{n\to\infty} y_n$ exists and coincides with $\inf\{y_n : n \in \mathbb{N}\}$.*

Proof: Let $x := \sup_{n\in\mathbb{N}}$. This is an upper bound of the sequence, hence $x_j \leqq x$ for all $j \in \mathbb{N}$. Let $\varepsilon > 0$. $x - \frac{\varepsilon}{2}$ is not an upper bound, hence there exists $N \in \mathbb{N}$ such that $x - \frac{\varepsilon}{2} \leqq x_N$. Because $(x_n)_n$ is non–decreasing, it follows for all $j \geqq N$ that

$$x - \varepsilon \ < \ x - \frac{\varepsilon}{2} \ \leqq \ x_N \ \leqq \ j \ \leqq \ x, \ \ \textit{hence } |x_j - x| \ = \ x - x_j \ < \ \varepsilon \textit{ for all } j \geqq N.$$

This proves that $x = \sup\limits_{n\in\mathbb{N}} \ = \ \lim\limits_{j\to\infty} x_j$. ■

Convergence is an extremely important concept in Mathematics, but it excludes the case of sequences such as $x_n := n$ and $y_n := -n$ ($n \in \mathbb{N}$). Intuition tells us that x_n converges to ∞ and y_n converges to $-\infty$ because we think of very big numbers as being very close to $+\infty$ and very small numbers (i.e., very big ones with a minus sign) as being very close to $-\infty$.

Definition 8.11 (Limit infinity)**.** For this definition we do not deal with an arbitrary metric space but specifically with $X = \mathbb{R}$ and $d(x,y) = |b - a|$. Given a real number $K > 0$, we define

(8.42a) $$N_K(\infty) := \{x \in \mathbb{R} : x > K\}$$

(8.42b) $$N_K(-\infty) := \{x \in \mathbb{R} : x < -K\}$$

We call $N_K(\infty)$ the K-neighborhood of ∞ and $N_K(-\infty)$ the K-neighborhood of $-\infty$.
We say that a sequence (x_n) has limit ∞ and we write either of

(8.43) $$x_n \to \infty \qquad \text{or} \qquad \lim_{n\to\infty} x_n = \infty$$

if the following is true for any (big) K: There is a (possibly extremely large) integer n_0 such that all x_j belong to $N_K(\infty)$ just as long as $j \geqq n_0$.
We say that the sequence (x_n) has limit $-\infty$ and we write either of

(8.44) $$x_n \to -\infty \qquad \text{or} \qquad \lim_{n\to\infty} x_n = -\infty$$

if the following is true for any (big) K: There is a (possibly extremely large) integer n_0 such that all x_j belong to $N_K(-\infty)$ just as long as $j \geqq n_0$. □

Note 8.1 (Notation for limits of monotone sequences)**.** Let (x_n) be a non-decreasing sequence of real numbers and let y_n be a non-increasing sequence. If $\xi = \lim_{k\to\infty} x_k$ (that limit might be $+\infty$) then we write suggestively

$$x_n \nearrow \xi \quad (n \to \infty)$$

If $\eta = \lim_{j\to\infty} y_j$ (that limit might be $-\infty$) then we write suggestively

$$y_j \searrow \eta \quad (j \to \infty) \ \square$$

We now briefly address continuity of functions which map real numbers to real numbers. This subject will be addressed in more detail and in more general settings in ch.10.2.1 on p.210. Let $A \subseteq \mathbb{R}$. Informally speaking, a continuous function $f : A \to \mathbb{R}$ is one whose graph in the xy–plane is a continuous line without any disconnections or gaps. This can be stated in slightly more formal terms by saying that, if the x-values are closely together then the $f(x)$-values must be closely together too.

Here is the formal definition.

Definition 8.12 (Continuity in $\mathbb{R}$)**.** Let $A \subseteq \mathbb{R}$, $x_0 \in A$, and let $f : A \to \mathbb{R}$ be a real–valued function with domain A. We say that f is **continuous at x_0** [68] and we write

$$\lim_{x\to x_0} f(x) = f(x_0) \tag{8.45}$$

if the following is true for **any** sequence (x_n) with values in A:

$$\text{if } x_n \to x_0 \text{ then } f(x_n) \to f(x_0). \tag{8.46}$$

In other words, the following must be true for any sequence (x_n) in A and $x_0 \in A$:

$$\lim_{n\to\infty} x_n = x_0 \quad \Rightarrow \quad \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x_0). \tag{8.47}$$

We say that f is **continuous** if f is continuous at x_0 for all $x_0 \in A$. $\square$

Remark 8.3. Important points to notice:

- **a)** It is not enough for the above to be true for some sequences that converge to x_0. Rather, it must be true for all such sequences!
- **b)** We restrict our universe to the domain A of f: x_0 and the entire sequence $(x_n)_{n\in\mathbb{N}}$ must belong to A because there must be function values for all x-values. $\square$

Proposition 8.12 (Rules of arithmetic for continuous real–valued functions with domain in $\mathbb{R}$)**.** *Let $A \in \mathbb{R}$. Assume that the functions*

$$f(\cdot), g(\cdot), f_1(\cdot), f_2(\cdot), f_3(\cdot), \dots, f_n(\cdot) : A \longrightarrow \mathbb{R}$$

all are continuous at $x_0 \in A$. Then

---

[68] We call such a function **sequence continous** in def.10.30 (Sequence continuity) on p.210 where continuity is generalized to metric spaces.

a. *Constant functions are continuous everywhere on* A.
b. *The product* $fg(\cdot) : x \mapsto f(x)g(x)$ *is continuous at* x_0. *Specifically,* $af(\cdot)x \mapsto a \cdot f(x)$ *is continuous at* x_0 *and, using* -1 *as a constant,* $-f(\cdot) : x \mapsto -f(x)$ *is continuous at* x_0.
c. *The sum* $f + g(\cdot) : x \mapsto f(x) + g(x)$ *is continuous at* x_0.
d. *Any linear combination* [69] $\sum\limits_{j=0}^{n} a_j f_j(\cdot) : x \mapsto \sum\limits_{j=0}^{n} a_j f_j(x)$ *is continuous in* x_0.

Proof: This proposition will be generalized in prop.10.24 on p.215 and a full proof will be given there. We give it here for the product fg *to demonstrate how knowledge about the convergence of sequences can be employed to prove statements concerning continuity.*

Let $x_n \in A$ *for all* $n \in \mathbb{N}$ *such that* $x_n \to x_0$ *as* $n \to \infty$. *All we need to show is* $f(x_n)g(x_n) \to f(x_0)g(x_0)$. *It follows from prop.8.10 (Rules of arithmetic for limits) on p.132 that* $\lim_{n\to\infty} f(x_n)g(x_n) = f(x_0)g(x_0)$. *This proves continuity of* $x \mapsto f(x)g(x) at x_0$. ■

A generalized version of the following theorem will be proved in a later chapter. [70]

Theorem 8.1. *Let* $A \subseteq \mathbb{R}$, $x_0 \in A$, *and let* $f : A \to \mathbb{R}$ *be a real–valued function with domain* A. *Then* f *is continuous at* x_0 *if and only if for any* $\varepsilon > 0$, *no matter how small, there exists* $\delta > 0$ *such that either one of the following equivalent statements is satisfied:*

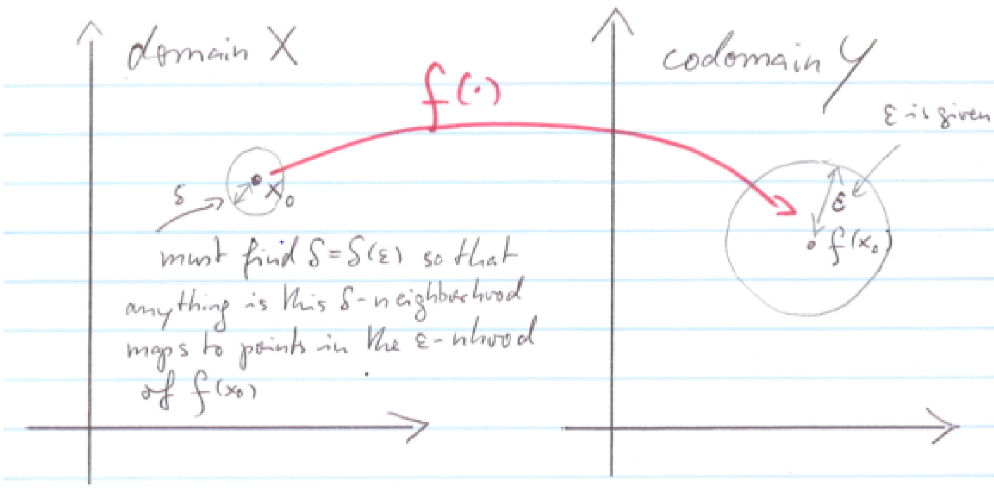
$$f\big(\{x \in A : |x - x_0| < \delta\}\big) \subseteq \{y \in \mathbb{R} : |f(x) - f(x_0)| < \varepsilon\}, \tag{8.48}$$

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \text{ for all } x \in A. \tag{8.49}$$

Proof: ■

[69] See def.9.6 (linear combinations) on p.160
[70] See thm.10.13 on p.211.

Figure 8.1: ε-δ continuity



8.3 Limit Inferior and Limit Superior

Definition 8.13 (Tail sets of a sequence)**.** Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Let

$$T_n \;:=\; \{x_j : j \geqq n\} \;=\; \{x_n, x_{n+1}, x_{n+2}, x_{n+3}, \dots\} \tag{8.50}$$

be what remains in the sequence after we discard the first $n-1$ elements. We call $(T_n)_{n\in\mathbb{N}}$ the **tail set** for n of the given sequence $(x_k)_{k\in\mathbb{N}}$. $\square$

Remark 8.4. Some simple properties of tail sets:

a. We deal with sets and not with sequences T_n: If, e.g., $x_n = (-1)^n$ then each $T_n = \{-1, 1\}$ only contains two items and not infinitely many.

b. The tail set sequence $(T_n)_{n\in\mathbb{N}}$ is "decreasing": If $m < n$ then $T_m \supseteq T_n$.

c. It follows from (b) and prop.8.3 on p.126 and prop.8.11 that

$$\beta_n \;:=\; \sup(T_n) \;\text{ is non–increasing, hence } \lim_{n\to\infty} \beta_n = \inf_n \beta_n;$$
$$\alpha_n \;:=\; \inf(T_n) \;\text{ is non–decreasing, hence } \lim_{n\to\infty} \alpha_n = \sup_n \alpha_n.$$

These limits can also be epressed as follows.

$$\begin{aligned} \lim_{n\to\infty} \big(\sup\{x_j : j \in \mathbb{N}, j \geqq n\}\big) \;&:=\; \lim_{n\to\infty} \big(\sup(T_n)\big) \;:=\; \inf\big(\{\sup(T_n) : n \in \mathbb{N}\}\big),\\ \lim_{n\to\infty} \big(\inf\{x_j : j \in \mathbb{N}, j \geqq n\}\big) \;&:=\; \lim_{n\to\infty} \big(\inf(T_n)\big) \;:=\; \sup\big(\{\inf(T_n) : n \in \mathbb{N}\}\big). \end{aligned} \tag{8.51}$$

An expression like $\sup\{x_j : j \in \mathbb{N}, j \geqq n\}$ can be written more compactly as $\sup\limits_{j\in\mathbb{N}, j\geqq n} \{x_j\}$. Moreover, when dealing with sequences (x_n), it is understood in most cases that $n \in \mathbb{N}$ or $n \in \mathbb{Z}_{\geq 0}$ and the last expression simplifies to $\sup\limits_{j\geqq n}\{x_j\}$. This can also be written as $\sup\limits_{j\geqq n}(x_j)$ or $\sup\limits_{j\geqq n} x_j$.

In other words, (8.51) becomes

$$(8.52)\qquad \begin{aligned} \inf_{n\in\mathbb{N}}\big(\sup_{j\geqq n} x_j\big) &= \inf\big(\{\sup(T_n) : n\in\mathbb{N}\}\big) = \lim_{n\to\infty}\big(\sup(T_n)\big) = \lim_{n\to\infty}\big(\sup_{j\geqq n} x_j\big),\\ \sup_{n\in\mathbb{N}}\big(\inf_{j\geqq n} x_j\big) &= \sup\big(\{\inf(T_n) : n\in\mathbb{N}\}\big) = \lim_{n\to\infty}\big(\inf(T_n)\big) = \lim_{n\to\infty}\big(\inf_{j\geqq n} x_j\big). \ \square \end{aligned}$$

The above leads us to the following definition:

Definition 8.14. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and let $T_n = \{x_j : j \in \mathbb{R}, j \geqq n\}$ be the tail set for x_n. Assume that T_n is bounded above for some $n_0 \in \mathbb{N}$ (and hence for all $n \geqq n_0$). We call

$$\limsup_{n\to\infty} x_j := \lim_{n\to\infty}\big(\sup_{j\geqq n} x_j\big) = \inf_{n\in\mathbb{N}}\big(\sup_{j\geqq n} x_j\big) = \inf_{n\in\mathbb{N}}\big(\sup(T_n)\big)$$

the **lim sup** or **limit superior** of the sequence (x_n).

If, for each n, T_n is not bounded above then we say $\limsup\limits_{n\to\infty} x_j = \infty$.

Assume that T_n is bounded below for some n_0 (and hence for all $n \geqq n_0$). We call

$$\liminf_{n\to\infty} x_j := \lim_{n\to\infty}\big(\inf_{j\geqq n} x_j\big) = \sup_{n\in\mathbb{N}}\big(\inf_{j\geqq n} x_j\big) = \sup_{n\in\mathbb{N}}\big(\inf(T_n)\big)$$

the **lim inf** or **limit inferior** of the sequence (x_n).

If, for each n, T_n is not bounded below then we say $\liminf\limits_{n\to\infty} x_j = -\infty$. $\square$

Proposition 8.13. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ which is bounded above with tail sets T_n.*

A. *Let*

$$(8.53)\qquad \begin{aligned} \mathscr{U} &:= \{y\in\mathbb{R} : T_n \cap [y,\infty[\neq \emptyset \text{ for all } n\in\mathbb{N}\},\\ \mathscr{U}_1 &:= \{y\in\mathbb{R} : \text{ for all } n\in\mathbb{N} \text{ there exists } k\in\mathbb{Z}_{\geq 0} \text{ such that } x_{n+k}\geqq y\},\\ \mathscr{U}_2 &:= \{y\in\mathbb{R} : \exists \text{ subsequence } n_1<n_2<n_3<\dots\in\mathbb{N} \text{ such that } x_{n_j}\geqq y \text{ for all } j\in\mathbb{N}\},\\ \mathscr{U}_3 &:= \{y\in\mathbb{R} : x_n \geqq y \text{ for infinitely many } n\in\mathbb{N}\}. \end{aligned}$$

Then $\mathscr{U} = \mathscr{U}_1 = \mathscr{U}_2 = \mathscr{U}_3$.

B. *There exists $z = z(\mathscr{U}) \in \mathbb{R}$ such that $\mathscr{U}$ is either an interval $]-\infty, z]$ or an interval $]-\infty, z[$.*

C. *Let $u := \sup(\mathscr{U})$. Then $u = z = z(\mathscr{U})$ as defined in part B. Further, u is the only real number such that*

- ***C1.*** (8.54) $\qquad u - \varepsilon \in \mathscr{U}$ *and* $u + \varepsilon \notin \mathscr{U}$ *for all* $\varepsilon > 0$.
- ***C2.*** *There exists a subsequence $(n_j)_{j\in\mathbb{N}}$ of integers such that $u = \lim\limits_{j\to\infty} x_{n_j}$ and u is the largest real number for which such a subsequence exists.*

*Proof of **A**:*

A.1 - $\mathscr{U} = \mathscr{U}_1$: This equality is valid by definition of tailsets of a sequence:

$$x \in T_n \;\Leftrightarrow\; x = x_j \text{ for some } j \geqq n \;\Leftrightarrow\; x = x_{n+k} \text{ for some } k \in \mathbb{Z}_{\geqq 0}$$

from which it follows that $x \in T_n \cap [y, \infty[\Leftrightarrow\; x = x_{n+k}$ for some $k \geqq 0$ and $x_{n+k} \geqq y$.

A.2 - $\mathscr{U}_1 \subseteq \mathscr{U}_2$:

Let $y \in \mathscr{U}_1$ and $n \in \mathbb{N}$. We prove the existence of $(n_j)_j$ by induction on j.

Base case $j = 1$: As $T_1 \cap [y, \infty[\neq \emptyset$ there is some $x \in T_1$ such that $y \leqq x < \infty$, i.e., $x \geqq y$. Because $x \in T_1 = \{x_1, x_2, \dots\}$ we have $x = x_{n_1}$ for some integer $n_1 \geqq 1$; we have proved the existence of n_1.

Induction assumption: Assume that $n_1 < n_2 < \cdots < n_{j_0}$ have already been picked.

Induction step: As $y \in \mathscr{U}_1$ there is $k \in \mathbb{Z}_{\geqq 0}$ such that $x_{(n_{j_0}+1)+k} \geqq y$. We set $n_{j_0+1} := n_{j_0} + 1 + k$. As this index is strictly larger than n_{j_0}, the induction step has been proved.

A.3 - $\mathscr{U}_2 \subseteq \mathscr{U}_3$: This is trivial: Let $y \in \mathscr{U}_2$. The strictly increasing subsequence $n_1 < n_2 < n_3 < \cdots \in \mathbb{N}$ constitutes the infinite set of indices that is required to grant y membership in $\mathscr{U}_3$.

A.4 - $\mathscr{U}_3 \subseteq \mathscr{U}$: Let $y \in \mathscr{U}_3$. Fix some $n \in \mathbb{N}$.

Let $J = J(y) \subseteq \mathbb{N}$ be the infinite set of indices j for which $x_j \geqq y$. At most finitely many of those j can be less than that given n and there must be (infinitely many) $j \in J$ such that $j \geqq n$

Pick any one of those, say j'. Then $x_{j'} \in T_n$ and $x_{j'} \geqq y$. It follows that $y \in \mathscr{U}$

We have shown the following sequence of inclusions:

$$\mathscr{U} \;=\; \mathscr{U}_1 \;\subseteq\; \mathscr{U}_2 \;\subseteq\; \mathscr{U}_3 \;\subseteq\; \mathscr{U}$$

It follows that all four sets are equal and part A of the proposition has been proved.

*Proof of **B**: Let $y_1, y_2 \in \mathbb{R}$ such that $y_1 < y_2$ and $y_2 \in \mathscr{U}$.*

It follows from $[y_2, \infty[\subseteq [y_1, \infty[$ that, because $T_n \cap [y_2, \infty[\neq \emptyset$ for all $n \in \mathbb{N}$, we must have $T_n \cap [y_1, \infty[\neq \emptyset$ for all $n \in \mathbb{N}$, i.e., $y_1 \in \mathscr{U}$.

But that means that $\mathscr{U}$ must be an interval of the form $]-\infty, z]$ or $]-\infty, z[$ for some $z \in \mathbb{R}$.

*Proof of **C**: Let $z = z(\mathscr{U})$ as defined in part B and $u := \sup(\mathscr{U})$.*

*Proof of **C.1 - (8.54) part 1**, $u - \varepsilon \in \mathscr{U}$:*

As $u - \varepsilon$ is smaller than the least upper bound u of $\mathscr{U}$, $u - \varepsilon$ is not an upper bound of $\mathscr{U}$. Hence there is $y > u - \varepsilon$ such that $y \in \mathscr{U}$. It follows from part B that $u - \varepsilon \in \mathscr{U}$.✓

*Proof of **C.1 - (8.54) part 2**, $u + \varepsilon \notin \mathscr{U}$:*

This is trivial as $u + \varepsilon > u = \sup(\mathscr{U})$ implies that $y \leqq u < u + \varepsilon$ for all $y \in \mathscr{U}$.

But then $y \neq u$ for all $y \in \mathscr{U}$, i.e., $u \notin \mathscr{U}$. This proves $u + \varepsilon \notin \mathscr{U}$.

*Proof of **C.2**: We construct by induction a sequence $n_1 < n_2 < \dots$ of natural numbers such that*

$$u \;-\; 1/j \;\leq x_{n_j} \;\leqq\; u \;+\; 1/j. \tag{8.55}$$

*Base case: We have proved as part of **C.1** that $x_n \geqq u+1$ for at most finitely many indices n. Let K be the largest of those.*

As $u-1 \in \mathscr{U}_3$, there are infinitely many n such that $x_n \geqq u-1$. Infinitely many of those n must exceed K. We pick one of them and that will be n_1. Clearly, n_1 satisfies (8.55) and this proves the base case.

Induction step: Let us now assume that $n_1 < n_2 < \cdots < n_k$ satisfying (8.55) have been constructed. $x_n \geqq u + 1/(k+1)$ is possible for at most finitely many indices n. Let K be the largest of those.

As $u-1/(k+1) \in \mathscr{U}_3$, there are infinitely many n such that $x_n \geqq u-1/(k+1)$. Infinitely many of those n must exceed $\max(K, n_k)$. We pick one of them and that will be n_{k+1}. Clearly, n_{k+1} satisfies (8.55) and this finishes the proof by induction.

We now show that $\lim\limits_{j\to\infty} x_{n_j} = u$. Given $\varepsilon > 0$ there is $N = N(\varepsilon)$ such that $1/N < \varepsilon$. It follows from (8.55) that $|x_{n_j} - u| \leqq 1/j < 1/N < \varepsilon$ for all $j \geqq n$ and this proves that $x_{n_j} \to u$ as $j \to \infty$.

*We will be finished with the proof of **C.2** if we can show that if $w > u$ then there is no sequence $n_1 < n_2 < \dots$ such that $x_{n_j} \to w$ as $j \to \infty$.*

Let $\varepsilon := (w-u)/2$. According to (8.54), $u + \varepsilon \notin \mathscr{U}$. But then, by definition of $\mathscr{U}$, there is $n \in \mathbb{N}$ such that $T_n \cap [u+\varepsilon, \infty[= \emptyset$.

But $u + \varepsilon = w - \varepsilon$ and we have $T_n \cap [w-\varepsilon, \infty[= \emptyset$. This implies that $|w - x_j| \geqq \varepsilon$ for all $j \geqq n$ and that rules out the possibility of finding n_j such that $\lim\limits_{j\to\infty} x_{n_j} = w$. ■

Corollary 8.2. *As in prop.8.13, let $u := \sup(\mathscr{U})$. Then $\mathscr{U} =]-\infty, u]$ or $\mathscr{U} =]-\infty, u[$.*

Further, u is determined by the following property: For any $\varepsilon > 0$, $x_n > u - \varepsilon$ for infinitely many n and $x_n > u + \varepsilon$ for at most finitely many n.

Proof: This follows from $\mathscr{U} = \mathscr{U}_3$ and parts B and C of prop.8.13. ■

When we form the sequence $y_n = -x_n$ then the roles of upper bounds and lower bounds, max and min, inf and sup will be reversed. Example: x is an upper bound for $\{x_j : j \geqq n$ if and only if $-x$ is a lower bound for $\{y_j : j \geqq n$.

The following "dual" version of prop. 8.13 is a direct consequence of the duality of upper/lower bounds, min/max, inf/sup proposition prop.8.2, p.125.

Proposition 8.14. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ which is bounded below with tail sets T_n.*

***A.** Let*

$$
\begin{aligned}
\mathscr{L} &:= \{y \in \mathbb{R} : T_n \cap]-\infty, y] \neq \emptyset \text{ for all } n \in \mathbb{N}\},\\
\mathscr{L}_1 &:= \{y \in \mathbb{R} : \text{ for all } n \in \mathbb{N} \text{ there exists } k \in \mathbb{Z}_{\geq 0} \text{ such that } x_{n+k} \leqq y\},\\
\mathscr{L}_2 &:= \{y \in \mathbb{R} : \exists \text{ subsequence } n_1 < n_2 < n_3 < \cdots \in \mathbb{N} \text{ such that } x_{n_j} \leqq y \text{ for all } j \in \mathbb{N}\},\\
\mathscr{L}_3 &:= \{y \in \mathbb{R} : x_n \leqq y \text{ for infinitely many } n \in \mathbb{N}\}.
\end{aligned}
\tag{8.56}
$$

Then $\mathscr{L} = \mathscr{L}_1 = \mathscr{L}_2 = \mathscr{L}_3$.

***B.** There exists $z = z(\mathscr{L}) \in \mathbb{R}$ such that $\mathscr{L}$ is either an interval $[z, \infty[$ or an interval $]z, \infty[$.*

C. *Let $l := \inf(\mathscr{L})$. Then $l = z = z(\mathscr{L})$ as defined in part B. Further, l is the only real number such that*

C1. (8.57) $$l + \varepsilon \in \mathscr{L} \quad \text{and} \quad l - \varepsilon \notin \mathscr{L}$$

C2. *There exists a subsequence $(n_j)_{j\in\mathbb{N}}$ of integers such that $l = \lim\limits_{j\to\infty} x_{n_j}$ and l is the smallest real number for which such a subsequence exists.*

Proof: Let $y_n = -x_n$ and apply prop.8.13. ■

Proposition 8.15. *Let (x_n) be a bounded sequence of real numbers. As in prop. 8.13 and prop 8.14, let*

(8.58) $$\begin{aligned} u &= \sup(\mathscr{U}) &= \sup\{y \in \mathbb{R} : T_n \cap [y, \infty[\neq \emptyset \text{ for all } n \in \mathbb{N}\}, \\ l &= \inf(\mathscr{L}) &= \inf\{y \in \mathbb{R} : T_n \cap]-\infty, y] \neq \emptyset \text{ for all } n \in \mathbb{N}\}, \end{aligned}$$

Then $\quad u = \limsup\limits_{n\to\infty} x_j \quad$ *and* $\quad l = \liminf\limits_{n\to\infty} x_j.$

Proof that $u = \limsup\limits_{n\to\infty} x_j$: Let

(8.59) $$\beta_n := \sup_{j \geqq n} x_j, \quad \beta := \inf_n \beta_n = \limsup_{n\to\infty} x_n.$$

We will prove that β has the properties listed in prop.8.13.C that uniquely characterize u: For any $\varepsilon > 0$, we have

$$\beta - \varepsilon \in \mathscr{U} \quad \text{and} \quad \beta + \varepsilon \notin \mathscr{U}$$

Another way of saying this is that

(8.60) $$b \in \mathscr{U} \text{ for } b < \beta \quad \text{and} \quad a \notin \mathscr{U} \text{ for } a > \beta.$$

We now prove the latter characterization.

Let $a \in \mathbb{R}, a > \beta = \inf\{\beta_n : n \in \mathbb{N}\}$. Then a is not a lower bound of the β_n: $\beta_{n_0} < a$ for some $n_0 \in \mathbb{N}$.

As the β_n are not increasing in n, this implies strict inequality $\beta_j < a$ for all $j \geqq n_0$. By definition, β_j is the least upper bound (hence an upper bound) of the tail set T_j. We conclude that $x_j < a$ for all $j \geqq n_0$.

From that we conclude that $T_n \cap [a, \infty[= \emptyset$ for all $j \geqq n_0$. It follows that $a \notin \mathscr{U}$.

Now let $b \in \mathbb{R}, b < \beta = g.l.b\{\beta_n : n \in \mathbb{N}\}$. As $\beta \leqq \beta_n$ we obtain $b < \beta_n$ for all n.

In other words, $b < \sup(T_n)$ for all n: It is possible to pick some $x_k \in T_n$ such that $b < x_k$.

But then $T_n \cap [b, \infty[\neq \emptyset$ for all n and we conclude that $b \in \mathscr{U}$.

We put everything together and see that β has the properties listed in (8.60). This finishes the proof that $u = \limsup\limits_{n\to\infty} x_j$. The proof that $l = \liminf\limits_{n\to\infty} x_j$ follows again by applying what has already been proved to the sequence $(-x_n)$. ■

We have collected everything to prove

Theorem 8.2 (Characterization of limsup and liminf). *Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}$. Then*

a1. $\limsup\limits_{n\to\infty} x_n$ *is the largest of all real numbers x for which a sequence $n_1 < n_2 < \cdots \in \mathbb{N}$ can be found such that $x = \lim\limits_{j\to\infty} x_{n_j}$.*

a2. $\limsup\limits_{n\to\infty} x_n$ *is the only real number u such that, for all $\varepsilon > 0$, the following is true:*
$x_n > u + \varepsilon$ for at most finitely many n and $x_n > u - \varepsilon$ for infinitely many n.

b1. $\liminf\limits_{n\to\infty} x_n$ *is the smallest of all real numbers x for which a sequence $n_1 < n_2 < \cdots \in \mathbb{N}$ can be found such that $x = \lim\limits_{j\to\infty} x_{n_j}$.*

b2. $\liminf\limits_{n\to\infty} x_n$ *is the only real number l such that, for all $\varepsilon > 0$, the following is true:*
$x_n < l - \varepsilon$ for at most finitely many n and $x_n < l + \varepsilon$ for infinitely many n.

*Proof: We know from prop.8.15 on p.141 that $\limsup\limits_{n\to\infty} x_n$ is the unique number u described in part **C** of prop.8.13, p.138:*

$u - \varepsilon \in \mathscr{U}$ *and* $u + \varepsilon \notin \mathscr{U}$ *for all* $\varepsilon > 0$

and u is the largest real number for which there exists a subsequence $(n_j)_{j\in\mathbb{N}}$ of integers such that $u = \lim\limits_{j\to\infty} x_{n_j}$.

*$u - \varepsilon \in \mathscr{U} = \mathscr{U}_3$ (see part A of prop.8.15) means that there are infinitely many n such that $x_n \geqq u - \varepsilon$ and $u + \varepsilon \notin \mathscr{U} = \mathscr{U}_3$ means that there are at most finitely many n such that $x_n \geqq u + \varepsilon$. This proves **a1** and **a2**.*

*We also know from prop.8.15 that $\liminf\limits_{n\to\infty} x_n$ is the unique number l described in part **C** of prop.8.14, p.140: $l + \varepsilon \in \mathscr{L}$ and $l - \varepsilon \notin \mathscr{L}$ for all $\varepsilon > 0$ and l is the smallest real number for which there exists a subsequence $(n_j)_{j\in\mathbb{N}}$ of integers such that $u = \lim\limits_{j\to\infty} x_{n_j}$.*

*$l + \varepsilon \in \mathscr{L} = \mathscr{L}_3$ (see part **A** of prop.8.15) means that there are infinitely many n such that $x_n \leqq l + \varepsilon$ and $l - \varepsilon \notin \mathscr{L} = \mathscr{L}_3$ means that there are at most finitely many n such that $x_n \leqq l - \varepsilon$. This proves **b1** and **b2**.*
■

*Proof of thm.8.2 **without the use of prop.8.15, prop.8.13 and the dual propositions for the liminf**.*

Step 1:

*Let $\varepsilon > 0$. It follows from $\beta_n = \sup(T_n) = \sup\{x_j : j \geqq n\}$ and $\beta_n \searrow \beta = \limsup_n x_n$ that $\beta_n < \beta + \varepsilon$ for all $n \geqq N$ for a suitable $N = N(\varepsilon) \in \mathbb{N}$. But then $\beta + \varepsilon$ exceeds the upper bound β_N of T_N and follows that all of its elements, i.e., all x_n with $n \geqq N$, satisfy $x_n < \beta + \varepsilon$. Hence only some or all of the finitely many $x_1, x_2, \dots x_{N-1}$ can exceed $\beta + \varepsilon$. It follows that β satisfies the first half of **a1** of thm.8.2.*

Step 2: We create a subsequence $(x_{n_j})_j$ such that

$$\beta_{n_j} \geqq x_{n_j} > \beta_{n_j} - 1/j \tag{8.61}$$

for all $j \in \mathbb{N}$ as follows.

$\beta_1 = \sup(T_1)$ is the smallest upper bound for T_1, hence $\beta_1 - 1$ is not an upper bound and we can find some $k \in \mathbb{N}$ such that $\beta_1 \geqq x_k > \beta_1 - 1$. We set $n_1 := k$.

Having constructed $n_1 < n_2 < \cdots < n_k$ such that $\beta_{n_j} \geqq x_{n_j} > \beta_{n_j} - 1/j$ for all $j \leqq k$ we now find $x_{n_{k+1}}$ with an index $n_{k+1} > n_k$ as follows.

$\beta_{n_k+1} - \frac{1}{k+1}$ is not an upper bound for T_{n_k+1}, hence there exists some $i \in \mathbb{N}$ such that x_{n_k+i} (which belongs to T_{n_k+1}) satisfies

$$x_{n_k+i} > \beta_{n_k+1} - \frac{1}{k+1}. \tag{8.62}$$

We set $n_{k+1} := n_k + i$. The sequence β_n non–increasing (i.e., decreasing) and it follows from $n_{k+1} = n_k + i \geqq n_k + 1$ that $\beta_{n_{k+1}} \leqq \beta_{n_k+1}$. But then (8.62) implies that $x_{n_{k+1}} > \beta_{n_{k+1}} - \frac{1}{k+1}$. We note that $x_{n_{k+1}} \leqq \beta_{n_{k+1}}$ because $x_{n_{k+1}} \in T_{n_{k+1}}$ and $\beta_{n_{k+1}} = \sup(T_{n_{k+1}})$ is an upper bound for all elements of $T_{n_{k+1}}$. Together with (8.62) we have

$$\beta_{n_{k+1}} \geqq x_{n_{k+1}} > \beta_{n_k+1} - \frac{1}{k+1}. \tag{8.63}$$

It follows that $x_{n_{k+1}}$ satisfies (8.61) and the the proof of step 1 is completed.

Step 3: The sequence x_{n_j} we constructed in step 2 converges to $\beta = \limsup_n x_n$. This is true because $\lim_k \beta_{n_k} = \beta$, $\lim_k \beta_{n_k} - 1/k = \lim_k \beta_{n_k} - \lim_k 1/k = \beta - 0 = \beta$ and x_{n_j} is "sandwiched" between two sequences which both converge to the same limit β.

*It follows from step 1 that no subsequence of (x_n) can converge to a number u bigger than β: Let $\varepsilon := 1/2(u - \beta)$. Then all but finitely many x_j satisfy $x_j \leqq \beta + \varepsilon$, hence $x_j \leqq u - \varepsilon$ and it follows that the distance $d(x_j, u)$ exceeds ε for $j \geqq N$ and no subsequence converging to u can be extracted. This proves **a1** of thm.8.2.*

*Step 4. We still must prove the missing half of thm.8.2.**a2**: $x_n > \beta - \varepsilon$ for infinitely many n.*

*Let $\varepsilon > 0$. and let $j \in \mathbb{N}$ be so big that $1/j < \varepsilon$. Let x_{n_j} be again the subsequence constructed in step 2. It follows from (8.61) and $\beta_{n_j} \geqq \beta$ and $1/j < \varepsilon$ that $x_{n_j} > \beta - \varepsilon$. This proves the missing half of thm.8.2.**a2**.*

*Uniqueness of β: If we have some $v > \beta$ tnen we set $\varepsilon := (v - \beta)/3$. Because $v - \varepsilon > \beta + \varepsilon$, at most finitely many x_n satisfy $x_n > v - \varepsilon$. It follows that v does not satisfy part 2 of thm.8.2.**a2**.*

*Finally let $v < \beta$. Let $\varepsilon := (\beta - v)/3$. Because $\beta - \varepsilon > v + \varepsilon$, infinitely many x_n satisfy $x_n > v + \varepsilon$. It follows that v does not satisfy part 1 of thm.8.2.**a2**. We have proved that $\limsup_n x_n$ is uniquely determined by the inequalities of thm.8.2.**a2** and we have shown both **a1** and **a2** of thm.8.2.*

*Parts **b1** and **b2** of thm.8.2 follow now easily from applying parts **a1** and **a2** to the sequence $y_n := -x_n$.* ■

Theorem 8.3 (Characterization of limits via limsup and liminf)**.** *Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}$. Then (x_n) converges to a real number if and only if liminf and limsup for that sequence coincide and we have*

$$\lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n. \tag{8.64}$$

Proof of "⇒": Let $L := \lim_{n\to\infty} x_n$. Let $\varepsilon > 0$. There is $N = N(\varepsilon) \in \mathbb{N}$ such that $T_k \subseteq \,]\, L - \varepsilon, L + \varepsilon \,[\,$ for all $k \geqq N$. But then

$$L - \varepsilon \leqq \alpha_k := \inf(T_k) \leq \beta_k := \sup(T_k) \leqq L + \varepsilon \;\; \text{for all } k \geqq N.$$

It follows from $T_j \subseteq T_k$ for all $j \geqq k$ that

$$L - \varepsilon \leq \alpha_k \leqq \alpha_j \leqq \beta_j \leqq \beta_k \leqq L + \varepsilon, \quad \textit{hence}$$
$$L - \varepsilon \leq \lim_{k\to\infty} \alpha_k = \liminf_{k\to\infty} x_k \leqq \limsup_{k\to\infty} x_k = \lim_{k\to\infty} \beta_k \leqq L + \varepsilon.$$

The equalities above result from prop.8.15. We have shown that, for any $\varepsilon > 0$, $\liminf_{k\to\infty} x_k$ and $\limsup_{k\to\infty} x_k$ differ by at most 2ε, hence they are equal.

Proof of "$\Leftarrow$": Let $L := \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$. Let $\varepsilon > 0$. We know from (8.54), p.138 and (8.57), p.141 that $L + \varepsilon/2 \notin \mathscr{U}$ and $L - \varepsilon/2 \notin \mathscr{L}$ But then there are at most finitely many n for which x_n has a distance from L which exceeds $\varepsilon/2$. Let N be the maximum of those n. It follows that $|x_n - L| < \varepsilon$ for all $n > N$, hence $L = \lim_{n\to\infty} x_n$. ■

Proposition 8.16. *Let $x_n, x'_n \in \mathbb{R}$ be two sequences of real numbers. Assume there is $K \in \mathbb{N}$ such that $x_n \leqq x'_n$ for all $n \geqq K$. Then*

$$\liminf_{n\to\infty} x_n \leqq \liminf_{n\to\infty} x'_n \qquad \textit{and} \qquad \limsup_{n\to\infty} x_n \leqq \limsup_{n\to\infty} x'_n.$$

Proof:

We only prove prove the limsup inequality because once we have that, we apply it to the sequences $(-x_n)_n$ and $(-x'_n)_n$ which satisfy $-x'_n \leqq -x_n$ for all $n \geqq K$. We obtain

$$-\liminf_{n\to\infty} x'_n = \limsup_{n\to\infty}(-x'_n) \leqq \limsup_{n\to\infty}(-x_n) = -\liminf_{n\to\infty} x_n,$$

hence $\liminf_{n\to\infty} x_n \leqq \liminf_{n\to\infty} x'_n$ and this proves the liminf inequality of the proposition.

***Case 1**: Both sequences are bounded.*

Let $u := \limsup_n x_n$ and $u' := \limsup_n x'_n$. We assume the contrary that $u > u'$. Then $\varepsilon := \frac{u-u'}{2} > 0$.

According to cor.8.2 on p.140 there are infinitely many $x_{n_1}, x_{n_2}, \dots$ such that $x_{n_j} > u - \varepsilon$. At most finitely of those n_j can be less than K. We discard those and there still are infinitely many $n_j \geqq K$ such that $x_{n_j} > u - \varepsilon$.

As $x'_i \geqq x_i$ for all $i \geqq K$, it follows that there are infinitely many n_j such that

$$x'_{n_j} \geqq x_{n_j} > u - \varepsilon = u' + \varepsilon.$$

We employ cor.8.2 a second time. It also states that there are at most finitely many x'_{n_j} such that $x'_{n_j} \geqq u' + \varepsilon$. We have reached a contradiction.

***Case 2**: Not both sequences are bounded above.*
If both are bounded below, $\liminf_n x_n \leqq \liminf_n x'_n$ is obtained just as in case 1, otherwise this is covered in case 3. We now observe what happens to the limits superior.

***Case 2a**: x_n is not bounded above.*
Then neither is x'_n and we that all tailsets for both sequences have sup $= \infty$, hence $\limsup_n x_n = \limsup_n x'_n = \infty$.

Case 2b: *x'_n is not bounded above.*
Then all tailsets for x'_n have sup = ∞, hence $\limsup_n x'_n = \infty$, hence $\limsup_n x_n \leqq \limsup_n x'_n$.

Case 3: *Not both sequences are bounded below.*
If both are bounded above, $\limsup_n x_n \leqq \limsup_n x'_n$ is obtained just as in case 1, otherwise this is covered in case 2. We now observe what happens to the limits inferior.

Case 2a: *x'_n is not bounded below.*
Then neither is x_n and we that all tailsets for both sequences have inf = $-\infty$, hence $\liminf_n x_n = \liminf_n x'_n = \infty$.

Case 2b: *x_n is not bounded above.*
Then all tailsets for x_n have inf = $-\infty$, hence $\liminf_n x_n = -\infty$, hence $\liminf_n x_n \leqq \liminf_n x'_n$.

■

Here is the first corollary to prop.8.16.

Corollary 8.3. *Let $x_n, y_n \in \mathbb{R}$ be two sequences of real numbers. Assume there is $K \in \mathbb{N}$ such that $x_n = y_n$ for all $n \geqq K$. Then*

$$\limsup_{n\to\infty} x_n = \limsup_{n\to\infty} y_n \quad \text{and} \quad \liminf_{n\to\infty} x_n = \liminf_{n\to\infty} y_n.$$

Proof:

■

Here is the second corollary to prop.8.16.

Corollary 8.4. *Let $x_n, y_n \in \mathbb{R}$ be two sequences of real numbers both of which have limits. Assume there is $K \in \mathbb{N}$ such that $x_n \leqq y_n$ for all $n \geqq K$. Then*

$$\lim_{n\to\infty} x_n \leqq \lim_{n\to\infty} y_n.$$

Proof:
■

8.4 Sequences of Sets and Indicator functions and their liminf and limsup

Let Ω be a non-empty set and let $f_n : \Omega \to \mathbb{R}$ be a sequence of real-valued functions. Let $\omega \in \Omega$. Then $\big(f_n(\omega)\big)_{n\in\mathbb{N}}$ is a sequence of real numbers for which we can examine $\liminf_n f_n(\omega)$ and $\limsup_n f_n(\omega)$. We will look at those two expressions as functions of ω.

Example 8.5. The following are examples of sequences of real–valued functions.

- **a.** $f_n : [0,1] \to \mathbb{R};\ x \mapsto x^n$ is a sequence of real–valued functions.
- **b.** Let $\Omega := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leqq 1\}$. be the unit circle in the Euclidean plane. Then $\varphi_n : \Omega \to \mathbb{R};\ \varphi_n(x,y) := \sqrt{x^2+y^2}$ is a sequence of real–valued functions.
- **c.** Let $f := \mathbb{R} \to \mathbb{R}$ be a (fixed, but arbitrary) function which is infinitely often differentiable at all its arguments, i.e., $D_n f(x_0) = f^{(n)}(x_0) = \frac{d^n f}{dx^n}\big|_{x=x_0}$ exists for all $x_0 \in \mathbb{R}$ and all $n \in \mathbb{N}$. Then $h_n : \mathbb{R} \to \mathbb{R};\ x \mapsto D_n f(x)$ is a sequence of real–valued functions. □

Definition 8.15 (limsup and liminf of a sequence of real functions)**.** Let Ω be a non-empty set and let $f_n : \Omega \to \mathbb{R}$ be a sequence of real-valued functions such that $f_n(\omega)$ is bounded for all $\omega \in \Omega$. [71]

We define

$$\liminf_{n\to\infty} f_n \ : \ \Omega \to \mathbb{R} \quad \text{as follows: } \ \omega \mapsto \liminf_{n\to\infty} f_n(\omega), \tag{8.66}$$

$$\limsup_{n\to\infty} f_n \ : \ \Omega \to \mathbb{R} \quad \text{as follows: } \ \omega \mapsto \limsup_{n\to\infty} f_n(\omega). \ \square \tag{8.67}$$

Remark 8.5. We recall from thm.8.2 (Characterization of limsup and liminf) on p.142 that

$$\liminf_{n\to\infty} f_n(\omega) \ = \inf\{\alpha \in \mathbb{R} : \ \lim_{j\to\infty} f_{n_j}(\omega) \ = \ \alpha \text{ for some subsequence } n_1 < n_2 < \dots\}, \tag{8.68}$$

$$\limsup_{n\to\infty} f_n(\omega) \ = \sup\{\beta \in \mathbb{R} : \ \lim_{j\to\infty} f_{n_j}(\omega) \ = \ \beta \text{ for some subsequence } n_1 < n_2 < \dots\}. \ \square \tag{8.69}$$

We now characterize $\liminf_n f_n$ *and* $\limsup_n f_n$ *for functions* f_n *such that* $f_n(\omega)$ *is either zero or one. We have seen in prop.6.12 on p.111 that any such function is the indicator function* 1_A *of the set*

$$A \ := \ \{f = 1\} \ = \ f^{-1}(\{1\}) \ = \ \{\omega \in \Omega : f(\omega) = 1\} \ \subseteq \ \Omega.$$

Proposition 8.17 (liminf and limsup of binary functions)**.** *Let* $\Omega \neq \emptyset$ *and* $f_n : \Omega \to \{0, 1\}$. *Let* $\omega \in \Omega$. *Then either* $\liminf_n f_n(\omega) \ = 1$ *or* $\liminf_n f_n(\omega) \ = 0$ *and either* $\limsup_n f_n(\omega) \ = 1$ *or* $\limsup_n f_n(\omega) \ = 0$*. Further*

$$\liminf_{n\to\infty} f_n(\omega) \ = 1 \ \Leftrightarrow f_n(\omega) \ = 1 \textbf{\textit{ except for at most finitely many }} n \in \mathbb{N} \tag{8.70}$$

$$\limsup_{n\to\infty} f_n(\omega) \ = 1 \ \Leftrightarrow f_n(\omega) \ = 1 \textit{ for infinitely many } n \in \mathbb{N} \tag{8.71}$$

Proof: It follows from (8.68), (8.69) *and* $0 \leqq f_n(\omega) \leqq 1$ *that* $0 \leqq \liminf_n f_n(\omega) \leqq \limsup_n f_n(\omega) \leqq 1$.

We conclude from (8.68) *that* $\liminf_n f_n(\omega) = 0$ *if a subsequence* $n_1 < n_2 < \dots$ *can be found such that* $f_{n_j}(\omega) = 0$ *for all* j *and that* $\liminf_n f_n(\omega) = 1$ *if no such subsequence exists, i.e., if* $f_n(\omega) = 1$ *for all except at most finitely many* n*. This proves not only (both directions(!) of)* (8.70) *but also that either* $\liminf_n f_n(\omega) \ = 1$ *or* $\liminf_n f_n(\omega) \ = 0$

We conclude from (8.69) *that* $\limsup_n f_n(\omega) = 1$ *if a subsequence* $n_1 < n_2 < \dots$ *can be found such that* $f_{n_j}(\omega) = 1$ *for all* j *and that* $\limsup_n f_n(\omega) = 0$ *if no such subsequence exists, i.e., if* $f_n(\omega) = 0$ *for*

[71] In more advanced texts you will find the following

Definition 8.16 (Extended real functions)**.** The set $\overline{\mathbb{R}} \ := \ \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ is called the **extended real numbers line**. A mapping F whose codomain is a subset of $\overline{\mathbb{R}}$ is called an **extended real function**. $\square$

The above allows to define the functions $\liminf_n f_n$ and $\limsup_n f_n$ even if there are arguments ω for which $\liminf_n f_n(\omega)$ and/or $\limsup_n f_n(\omega)$ assumes one of the values $\pm\infty$. There are many issues with functions that allow some arguments to have infinite value (hint: if $F(x) = \infty$ and $F(y) = \infty$, what is $F(x) - F(y)$?)

We only list the following rule which might come unexpected to you:

$$0 \cdot \pm\infty \ = \ \pm\infty \cdot 0 \ = \ 0 \tag{8.65}$$

all except at most finitely many n. This proves not only (both directions(!) of) (8.71) but also that either $\limsup_n f_n(\omega) = 1$ *or* $\limsup_n f_n(\omega) = 0$. ■

We now look at indicator functions 1_{A_n} of a sequence of sets $A_n \subseteq \Omega$. For such a sequence we define

$$A_\star := \bigcup_{n\in\mathbb{N}} \bigcap_{j\geqq n} A_j, \qquad A^\star := \bigcap_{n\in\mathbb{N}} \bigcup_{j\geqq n} A_j. \tag{8.72}$$

Proposition 8.18. *Let $\omega \in \Omega$. Then*

$$\omega \in A_\star \Leftrightarrow \omega \in A_n \text{ for all except at most finitely many } n \in \mathbb{N}. \tag{8.73}$$

$$\omega \in A^\star \Leftrightarrow \omega \in A_n \text{ for infinitely many } n \in \mathbb{N}, \tag{8.74}$$

a. *Proof that $\omega \in A_\star \Rightarrow \omega \in A_n$ for all except at most finitely many $n \in \mathbb{N}$:*
We will prove the contrapositive: Assume that there exists $1 \leqq n_1 < n_2 < \dots$ such that $\omega \notin A_{n_j}$ for all $j \in \mathbb{N}$. We must show that $\omega \notin A_\star$.

Let $k \in \mathbb{N}$. Then $k \leqq n_k$ (think!) and it follows from $\omega \notin A_{n_k}$ and $A_{n_k} \supseteq \bigcap_{j\geqq n_k} A_j \supseteq \bigcap_{j\geqq k} A_j$ that there is no $k \in \mathbb{N}$ such that $\omega \in \bigcap_{j\geqq k} A_j$.

*But then $\omega \notin \bigcup_k \bigcap_{j\geqq k} A_j = A_\star$ and we are done with the proof of **a**.*

b. *Proof that $\omega \in A_n$ for all except at most finitely many $n \in \mathbb{N} \Rightarrow \omega \in A_\star$:*
By assumption there exists some $N \in \mathbb{N}$ such that $\omega \in A_n$ for all $n \geqq N$.

*It follows that $\omega \in \bigcap_{n\geqq N} A_n \subseteq \bigcup_{m\in\mathbb{N}} \bigcap_{n\geqq m} A_n = A_\star$ and **b** has been proved.*

c. *Proof that $\omega \in A^\star \Rightarrow \omega \in A_n$ for infinitely many $n \in \mathbb{N}$:*
Let $\omega \in A^\star$. We will recursively construct $1 \leqq n_1 < n_2 < \dots$ such that $\omega \in A_{n_j}$ for all $j \in \mathbb{N}$.

We observe that $\omega \in \bigcup_{j\geqq n} A_j$ for all $n \in \mathbb{N}$. As $\omega \in \bigcup_{j\geq 1} A_j$ there exists $n_1 \geqq 1$ such that $\omega \in A_{n_1}$ and we have constructed the base case.

*Let $k \in \mathbb{N}$. If we already have found $n_1 < n_2 < \dots n_k$ such that $\omega \in A_{n_j}$ for $1 \leqq j \leqq k$ then we find n_{k+1} as follows: As $\omega \in \bigcup_{j\geqq n_k+1} A_j$ there exists $n_{k+1} \geqq n_k + 1$ such that $\omega \in A_{n_{k+1}}$. We have constructed our infinite sequence and this finishes the proof of **c**.*

d. *Proof that if $\omega \in A_n$ for infinitely many $n \in \mathbb{N} \Rightarrow \omega \in A^\star$:*
For $n \in \mathbb{N}$ we abbreviate $\Gamma_n := \bigcup_{j\geqq n} A_j$.

Let $1 \leqq n_1 < n_2 < \dots$ such that $\omega \in A_{n_j}$ for all $j \in \mathbb{N}$. Let $k \in \mathbb{N}$.

*Then $n_k \geqq k$, hence $\omega \in A_{n_k} \in \Gamma_{n_k} \subseteq \Gamma_k$ for all $k \in \mathbb{N}$, hence $\omega \in \bigcap_{k\in\mathbb{N}} \Gamma_k = A^\star$. We have proved **d**.* ■

Proposition 8.19 (liminf and limsup of indicator functions).

$$1_{A_\star} = \liminf_{n\to\infty} 1_{A_n} \quad \textit{and} \quad 1_{A^\star} = \limsup_{n\to\infty} 1_{A_n} \tag{8.75}$$

Proof: Let $\omega \in \Omega$. Then

$$1_{A_\star}(\omega) = 1 \;\Leftrightarrow\; \omega \in A_\star \;\Leftrightarrow \omega \in A_n \text{ for all except at most finitely many } n \in \mathbb{N}$$
$$\Leftrightarrow 1_{A_n}(\omega) = 1 \text{ for all except at most finitely many } n \in \mathbb{N} \tag{8.76}$$
$$\Leftrightarrow \liminf_n 1_{A_n}(\omega) = 1$$

The second equivalence follows from prop.8.18 and the last equivalence follows from prop.8.17 and this proves the first equation. Similarly we have

$$1_{A^\star}(\omega) = 1 \;\Leftrightarrow\; \omega \in A^\star \;\Leftrightarrow \omega \in A_n \text{ for infinitely many } n \in \mathbb{N}$$
$$\Leftrightarrow 1_{A_n}(\omega) = 1 \text{ for infinitely many } n \in \mathbb{N} \tag{8.77}$$
$$\Leftrightarrow \limsup_n 1_{A_n}(\omega) = 1$$

Again the second equivalence follows from prop.8.18 and the last equivalence follows from prop.8.17. ■

This last proposition is the reason for the following definition.

Definition 8.17 (limsup and liminf of a sequence of sets)**.** Let Ω be a non-empty set and let $A_n \subseteq \Omega$ $(n \in \mathbb{N})$. We define

$$\liminf_{n\to\infty} A_n := \bigcup_{n\in\mathbb{N}} \bigcap_{j\geqq n} A_j, \tag{8.78}$$

$$\limsup_{n\to\infty} A_n := \bigcap_{n\in\mathbb{N}} \bigcup_{j\geqq n} A_j. \tag{8.79}$$

We call $\liminf_{n\to\infty} A_n$ the **limit inferior** and $\limsup_{n\to\infty} A_n$ the **limit superior** of the sequence A_n.

We note that $\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$ if and only if the functions $\liminf_{n\to\infty} 1_{A_n}$ and $\limsup_{n\to\infty} 1_{A_n}$ coincide (prop. 8.19) which is true if and only if the sequence $1_{A_n}(\omega)$ has a limit for all $\omega \in \Omega$ (thm.8.3 on p.143). In this case we define

$$\lim_{n\to\infty} A_n := \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n \tag{8.80}$$

and we call this set the **limit** of the sequence A_n. □

Note 8.2 (Notation for limits of monotone sequences of sets)**.** Let (A_n) be a non-decreasing sequence of sets, i.e., $A_1 \subseteq A_2 \subseteq \dots$ and let $A := \bigcup_n A_n$. Further let B_n be a non-increasing sequence of sets, i.e., $B_1 \supseteq B_2 \supseteq \dots$ and let $B := \bigcap_n B_n$. We write suggestively [72]

$$A_n \nearrow A \quad (n \to \infty), \qquad B_n \searrow B \quad (n \to \infty). \;\square$$

Example 8.6. Let $A_n \subseteq \Omega$.

$$\textbf{a. } \text{If } A_n \nearrow \text{ then } \quad \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} A_n. \tag{8.81}$$

$$\textbf{b. } \text{If } A_n \searrow \text{ then } \quad \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} A_n. \;\square \tag{8.82}$$

[72] See note 8.1 on p.134.

Exercise 8.2. Prove the assertions of example 8.6 above. $\square$

Note 8.3 (Liminf and limsup of number sequences vs their tail sets)**.** Let $x_n \in \mathbb{R}$ be a sequence of real numbers. We then can associate with this sequence that of its tail sets $T_n := \{x_j : j \geqq n\}$.

a. Do not confuse $\liminf\limits_n x_n = \sup\limits_n \left(\inf(T_n)\right)$ with $\liminf\limits_n T_n = \bigcup\limits_n \left(\bigcap\limits_{k \geq n} T_k\right)$.

b. Do not confuse $\limsup\limits_n x_n = \inf\limits_n \left(\sup(T_n)\right)$ with $\limsup\limits_n T_n = \bigcap\limits_n \left(\bigcup\limits_{k \geq n} T_k\right)$.

Those two concepts are very different: $\liminf\limits_n x_n$ ($\limsup\limits_n x_n$) is a number: it is the lowest possible (highest possible) limit of a convergent subsequence $(x_{n_j})_{j \in \mathbb{N}}$. On the other hand we deal with a set(!) $\liminf_n T_n = \limsup_n T_n = \bigcap_n T_n$. The last equality follows from example 8.6 and the fact the the sequence of tailsets T_n is always non-increasing. $\square$

8.5 Sequences that Enumerate Parts of $\mathbb{Q}$

We informally defined the real numbers in ch.2.2 (Numbers) on p.15 as the set of all decimals, i.e., all numbers x which can be written as

$$x = m + \sum_{j=1}^{\infty} d_j 10^{-j} \quad \text{where } d_j \text{ is a digit, i.e., } d_j = 0, 1, 2, \dots, 9, \tag{8.83}$$

$$\text{i.e., } x = \lim_{k \to \infty} x_k \quad \text{where } x_k = m + \sum_{j=1}^{k} d_j 10^{-j}. \tag{8.84}$$

Each x_k is a (finite) sum of fractions, hence $x_k \in \mathbb{Q}$.

*We proved in cor.**??** on p.**??** that $\mathbb{Q}$ and hence all of its subsets can be sequenced: If $A \subseteq \mathbb{Q}$ there is a sequence $(q_n)_n$ of fractions such that $A = \{q_n : n \subseteq \mathbb{N}\}$. We apply this to $A := \mathbb{Q}$ as follows.*

Let $x \in \mathbb{R}$ have the representation (8.84). Then $x_k \in \mathbb{Q}$ for each $k \in \mathbb{N}$, hence there is some $n \in \mathbb{N}$ such that $x_k = q_n$. Of course n depends on k, i.e., we have a functional dependency $n = n(k) = n_k$. It follows from (8.84) that $q_{n_k} \to x$ as $k \to \infty$. In other words, we have proved the following

Theorem 8.4 (Universal sequence of rational numbers with convergent subsequences to any real number)**.**

There is a sequence $(q_n)_{n \in \mathbb{N}}$ of fractions which satisfies the following: For any $x \in \mathbb{R}$ there is a sequence $n_1, n_2, n_3, \dots,$ of natural numbers such that $x = \lim\limits_{k \to \infty} q_{n_k}$ ■.

Remark 8.6.

a. The above theorem can be phrased as follows: There is a sequence $(q_n)_{n \in \mathbb{N}}$ of fractions such that for any $x \in \mathbb{R}$ one can find a subsequence $(q_{n_j})_{j \in \mathbb{N}}$ of $(q_n)_n$ which converges to x.

b. What is remarkable about thm.8.4: A **single** sequence $(q_n)_n$ is so rich that its ingredients can be used to approximate any item in the uncountable**!** set $\mathbb{R}$

c. Let $A := \{x \in \mathbb{R} : x^2 \leqq 2\} \; = [\,-\sqrt{2}, \sqrt{2}\,]$ and let $A_{\mathbb{Q}} := A \cap \mathbb{Q} = \{q \in \mathbb{Q} : q^2 \leqq 2\}$. A is of such a shape that for any $x \in A$ the partial sums $x_k \;=\; m \;+\; \sum_{j=1}^{k} d_j 10^{-j}$ which converge to x belong to $A_{\mathbb{Q}}$. (Why? Especially, why also for $x = \pm\sqrt{2}$?) $\square$

8.6 Exercises for Ch.8

Exercise 8.3. Prove prop.8.7 on p.130 for the case that $\lim\limits_{n\to\infty} x_n$ exists in $\mathbb{R}$:
Let $(x_n)_n$ be a sequence of real numbers such that $\lim_{n\to\infty} x_n$ exists. Let $K \in \mathbb{N}$. For $n \in \mathbb{N}$ let $y_n := x_{n+K}$. Then $(y_n)_n$ has the same limit. $\square$

Exercise 8.4. Prove cor.8.3 on p.145:
Let $x_n, y_n \in \mathbb{R}$ be two sequences of real numbers. Assume there is $K \in \mathbb{N}$ such that $x_n = y_n$ for all $n \geqq K$. Then

$$\limsup_{n\to\infty} x_n = \limsup_{n\to\infty} y_n \quad \text{and} \quad \liminf_{n\to\infty} x_n = \liminf_{n\to\infty} y_n. \;\square$$

Exercise 8.5. Prove cor.8.4 on p.145:
Let $x_n, y_n \in \mathbb{R}$ be two sequences of real numbers both of which have limits. Assume there is $K \in \mathbb{N}$ such that $x_n \leqq y_n$ for all $n \geqq K$. Then

$$\lim_{n\to\infty} x_n \leqq \lim_{n\to\infty} y_n. \;\square$$

9 Vectors and Vector spaces (Understand this!)

9.1 $\mathbb{R}^N$: Euclidean Space

Most if not all of the material of this chapter with the exception of ch.9.2.2 (Normed Vector Spaces) on p.164 is familiar to anyone who took a linear algebra course or, in case of two or three dimensional space, to those who took a course in multivariable calculus.

9.1.1 N–Dimensional Vectors

This following definition of a vector is much more specialized than what is usually understood among mathematicians. For them, a vector is an element of a ***"vector space"*** *. You can find later in the document the definition of a vector space ((9.4) on p.157) What you see here is a definition of vectors of "finite dimension".*

Definition 9.1 (N–dimensional vectors)**.** A **vector** is a finite, ordered collection $\vec{v} = (x_1, x_2, x_3, \dots, x_N)$ of real numbers $x_1, x_2, x_3, \dots, x_N$. "Ordered" means that it matters which number comes first, second third, . . . If the vector has N elements then we say that it is N**–dimensional** . The set of all N–dimensional vectors is written as $\mathbb{R}^N$. $\square$

You are encouraged to go back to the section on cartesian products (5.4 on p.99) to review what was said there about $\mathbb{R}^N = \underbrace{\mathbb{R} \times \mathbb{R} \times + \cdots \times \mathbb{R}}_{N times}$. *Here are some examples of vectors:*

Example 9.1 (Two–dimensional vectors)**.** The two–dimensional vector with coordinates $x = -1.5$ and $y = \sqrt{2}$ is written $(-1.5, \sqrt{2})$ and we have $(-1.5, \sqrt{2}) \in \mathbb{R}^2$. Order matters, so this vector is different from $(\sqrt{2}, -1.5) \in \mathbb{R}^2$. $\square$

Example 9.2 (Three–dimensional vectors)**.** $\vec{v_t} = (3-t, 15, \sqrt{5t^2 + \frac{22}{7}}) \in \mathbb{R}^3$ with coordinates $x = 3-t$, $y = 15$ and $z = \sqrt{5t^3 + \frac{22}{7}}$ is an example of a parametrized vector (parametrized by t). Each specific value of t defines an element of $\in \mathbb{R}^3$, e.g., $\vec{v}_{-2} = (5, 15, \sqrt{20 + \frac{22}{7}})$. Note that

$$F : \mathbb{R} \to \mathbb{R}^3 \qquad t \mapsto F(t) = \vec{v_t}$$

defines a mapping from $\mathbb{R}$ into $\mathbb{R}^3$ in the sense of definition (4.6) on p.75. Each argument s has assigned to it one and only one argument $\vec{v_s} = (3 - s, 15, \sqrt{5s^2 + \frac{22}{7}}) \in \mathbb{R}^3$.
Or, is it rather that we have three functions

$$\begin{array}{ll} x(\cdot) : \mathbb{R} \to \mathbb{R} & t \to x(t) = 3 - t. \\ y(\cdot) : \mathbb{R} \to \mathbb{R} & t \to y(t) = 15. \\ z(\cdot) : \mathbb{R} \to \mathbb{R} & t \to z(t) = \sqrt{5t^2 + \frac{22}{7}} \end{array}$$

and $t \to \vec{v_t} = (x(t), y(t), z(t))$ is a vector of three real valued functions $x(\cdot), y(\cdot), z(\cdot)$?

Both points of view are correct and it depends on the specific circumstances in which way you want to interpret $\vec{v_t}$. $\square$

Example 9.3 (One–dimensional vectors)**.** Let us not forget about the one–dimensional case: A one-dimensional vector has a single coordinate.

For example, $\vec{w_1} = (-3) \in \mathbb{R}^1$ with coordinate $x = -3 \in \mathbb{R}$ and $\vec{w_2} = (5.7a) \in \mathbb{R}^1$ with coordinate $x = 5.7a \in \mathbb{R}$ are one–dimensional vectors. $\vec{w_2}$ is not a fixed number but parametrized by a.

Mathematicians do not distinguish between the one–dimensional vector (x) and its coordinate value, the real number x. For brevity, they will simply write $\vec{w_1} = -3$ and $\vec{w_2} = 5.7a$. $\square$

Example 9.4 (Vectors as functions)**.** An N–dimensional vector $\vec{x} = (x_1, x_2, x_3, \cdots, x_N)$ can be interpreted as a real function (remember: a real function is one which maps it arguments into $\mathbb{R}$)

$$\begin{aligned} &f_{\vec{x}}(\cdot) : \{1, 2, 3, \cdots, N\} \to \mathbb{R} \qquad m \mapsto x_m \\ &f_{\vec{x}}(1) = x_1,\ f_{\vec{x}}(2) = x_2,\ \cdots,\ f_{\vec{x}}(N) = x_N, \end{aligned} \tag{9.1}$$

i.e., as a real function whose domain is the natural numbers $1, 2, 3, \cdots, N$. This goes also the other way around: given a real function $f(\cdot) : \{1, 2, 3, \cdots, N\} \to \mathbb{R}$ we can associate with it the vector

$$\begin{aligned} &\vec{v}_{f(\cdot)} := (f(1), f(2), f(3), \cdots, f(N)) \\ &\vec{v}_{f_1} = f(1),\ \vec{v}_{f_2} = f(2),\ , \cdots, \vec{v}_{f_N} = f(N)\ \square \end{aligned} \tag{9.2}$$

9.1.2 Addition and Scalar Multiplication for N–Dimensional Vectors

Definition 9.2 (Addition and scalar multiplication in $\mathbb{R}^N$)**.** Given are two N–dimensional vectors $\vec{x} = (x_1, x_2, \dots, x_N)$ and $\vec{y} = (y_1, y_2, \dots, y_N)$ and a real number α.

We define the **sum** $\vec{x} + \vec{y}$ of $\vec{x}$ and $\vec{y}$ as the vector $\vec{z}$ with the components

$$z_1 = x_1 + y_1; \quad z_2 = x_2 + y_2; \quad \dots; \quad z_N = x_N + y_N; \tag{9.3}$$

We define the **scalar product** $\alpha\vec{x}$ of α and $\vec{x}$ as the vector $\vec{w}$ with the components

$$w_1 = \alpha x_1; \quad w_2 = \alpha x_2; \quad \dots; \quad w_N = \alpha x_N.\ \square \tag{9.4}$$

Figure 9.1 below describes vector addition.

Adding two vectors $\vec{v}$ and $\vec{w}$ means that you take one of them, say $\vec{v}$, and shift it in parallel (without rotating it in any way or flipping its direction), so that its starting point moves from the origin to the endpoint of the other vector $\vec{w}$. Look at the picture and you see that the vectors $\vec{v}$, $\vec{w}$ and $\vec{v}$ shifted form three pages of a parallelogram. $\vec{v} + \vec{w}$ is then the diagonal of this parallelogram which starts at the origin and ends at the endpoint of $\vec{v}$ shifted.

9.1.3 Length of N–Dimensional Vectors and the Euclidean Norm

*It is customary to write $\|\vec{v}\|_2$ for the length, sometimes also called the **Euclidean norm** of the vector $\vec{v}$.*

Example 9.5 (Length of one–dimensional vectors)**.** For a vector $\vec{v} = x \in \mathbb{R}$ its length is its absolute value $\|\vec{v}\|_2 = |x|$. This means that $\|-3.57\|_2 = |-3.57| = 3.57$ and $\|\sqrt{2}\|_2 = |\sqrt{2}| \approx 1.414$. $\square$

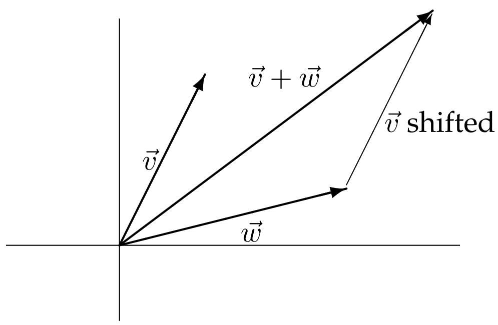


Figure 9.1: Adding two vectors.

Example 9.6 (Length of two–dimensional vectors)**.** We start with an example. Look at $\vec{v} = (4, -3)$. Think of an xy-coordinate system with origin (the spot where x-axis and y-axis intersect) $(0, 0)$. Then $\vec{v}$ is represented by an arrow which starts at the origin and ends at the point with coordinates $x = 4$ and $y = -3$ (see figure 9.2). How long is that arrow?

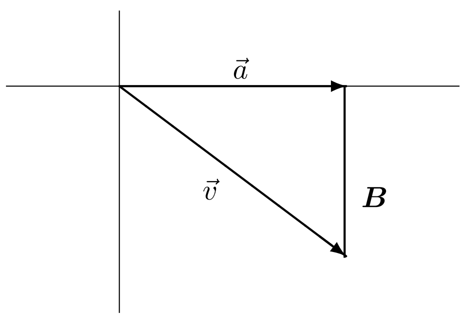


Figure 9.2: Length of a 2–dimensional vectors.

Think of it as the hypothenuse of a right angle triangle whose two other sides are the horizontal arrow from $(0, 0)$ to $(4, 0)$ (the vector $\vec{a} = (4, 0)$) and the vertical line $\boldsymbol{B}$ between $(4, 0)$ and $(4, -3)$. Note that $\boldsymbol{B}$ is not a vector because it does not start at the origin! Obviously (I hope this is obvious) we have $\|\vec{a}\|_2 = 4$ and length–of$(\boldsymbol{B}) = 3$. Pythagoras tells us that

$$\|\vec{v}\|_2^2 \;=\; \|\vec{a}\|_2^2 + \big(\text{length–of–}\boldsymbol{B}\big)^2$$

and we obtain for the vector $(4, -3)$ that $\|\vec{v}\|_2 = \sqrt{16 + 9} = 5$.

The above argument holds for any vector $\vec{v} = (x, y)$ with arbitrary $x, y \in \mathbb{R}$. The horizontal leg on the x-axis is then $\vec{a} = (x, 0)$ with length $|x| = \sqrt{x^2}$ and the vertical leg on the y-axis is a line equal in length to $\vec{b} = (0, y)$ the length of which is $|y| = \sqrt{y^2}$ The theorem of Pythagoras yields $\|(x,y)\|_2^2 = x^2 + y^2$ which becomes, after taking square roots on both sides,

$$\|(x,y)\|_2 = \sqrt{x^2 + y^2} \quad \square \tag{9.5}$$

Example 9.7 (Length of three–dimensional vectors)**.** This is not so different from the two–dimensional case above. We build on the previous example. Let $\vec{v} = (4, -3, 12)$. Think of an xyz-coordinate system with origin (the spot where x-axis, y-axis and z-axis intersect) $(0,0,0)$. Then $\vec{v}$ is represented by an arrow which starts at the origin and ends at the point with coordinates $x = 4$, $y = -3$ and $z = 12$. How long is that arrow?

Remember what the standard 3–dimensional coordinate system looks like: The x-axis goes from west to east, the y-axis goes from south to north and the z-axis goes vertically from down below to the sky. Now drop a vertical line $\boldsymbol{B}$ from the point with coordinates $(4, -3, 12)$ to the xy–plane which is "spanned" by the x-axis and y-axis. This line will intersect the xy–plane at the point with coordinates $x = 4$ and $y = -3$ (and $z = 0$. Why?)

Note that $\boldsymbol{B}$ is not a vector because it does not start at the origin! It should be clear that length–of$(\boldsymbol{B}) = |z| = 12$.

Now we connect the origin $(0,0,0)$ with the point $(4, -3, 0)$ in the xy–plane which is the endpoint of $\boldsymbol{B}$.

We can forget about the z–dimension because this arrow is entirely contained in the xy–plane. Matter of fact, it is a genuine two–dimensional vector $\vec{a} = (4, -3)$ because it starts in the origin. Observe that $\vec{a}$ has the same values 4 and -3 for its x– and y–coordinates as the original vector $\vec{v}$. [73] We know from the previous example about two–dimensional vectors that

$$\|\vec{a}\|_2^2 = \|(x,y)\|_2^2 = x^2 + y^2 = 16 + 9 = 25.$$

At this point we have constructed a right angle triangle with a) hypothenuse $\vec{v} = (x, y, z)$ where we have $x = 4$, $y = -3$ and $z = 12$, b) a vertical leg with length $|z| = 12$ and c) a horizontal leg with length $\sqrt{x^2 + y^2}$ = 5. Pythagoras tells us that

$$\|\vec{v}\|_2^2 = z^2 + \|(x,y)\|_2^2 \ = \ 144 + 25 = 169 \quad \text{or} \quad \|\vec{v}\|_2 = 13.$$

None of what we just did depended on the specific values 4, -3 and 12. Any vector $(x, y, z) \in \mathbb{R}^3$ is the hypothenuse of a right triangle where the square lengths of the legs are z^2 and $x^2 + y^2$. This means we have proved the general formula $\|(x,y,z)\|^2 = x^2 + y^2 + z^2$ or

$$\|(x,y,z)\| = \sqrt{x^2 + y^2 + z^2} \quad \square \tag{9.6}$$

The previous examples show how to extend the concept of "length" to vector spaces of any finite dimension:

Definition 9.3 (Euclidean norm)**.** Let $n \in \mathbb{N}$ and $\vec{v} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ be an n–dimension vector. The **Euclidean norm** $\|\vec{v}\|_2$ of $\vec{v}$ is defined as follows:

$$\|\vec{v}\|_2 \quad = \quad \sqrt{{x_1}^2 + {x_2}^2 + \dots + {x_n}^2} \quad = \quad \sqrt{\sum_{j=1}^{n} {x_j}^2}. \ \square \tag{9.7}$$

[73] You will learn in the chapter on vector spaces that the vector $\vec{a} = (4, -3)$ is the projection on the xy–coordinates $\pi_{1,2}(\cdot) : \mathbb{R}^3 \to \mathbb{R}^2 \qquad (x,y,z) \mapsto (x,y)$ of the vector $\vec{v} = (4, -3, 12)$. (see Example 9.19) on p.163)

This definition is important enough to write the special cases for $n = 1, 2, 3$ where $\|\vec{v}\|_2$ coincides with the length of $\vec{v}$:

$$(9.8)\qquad \begin{aligned} 1-dim: \quad \|(x)\|_2 &= \sqrt{x^2} = |x| \\ 2-dim: \quad \|(x,y)\|_2 &= \sqrt{x^2+y^2} \\ 3-dim: \quad \|(x,y,z)\|_2 &= \sqrt{x^2+y^2+z^2} \end{aligned}$$

Proposition 9.1 (Properties of the Euclidean norm)**.** *Let $n \in \mathbb{N}$. Then the Euclidean norm, viewed as a function*

$$\|\cdot\|_2: \quad \mathbb{R}^n \longrightarrow \mathbb{R} \qquad \vec{v} = (x_1, x_2, \dots, x_n) \longmapsto \|\vec{v}\|_2 = \sqrt{\sum_{j=1}^{n} {x_j}^2}$$

has the following three properties:

(9.9a)	$\|\vec{v}\|_2 \geqq 0 \quad \forall \vec{v} \in \mathbb{R}^n$ *and* $\|\vec{v}\|_2 = 0 \Leftrightarrow \vec{v} = 0$	***positive definiteness***
(9.9b)	$\|\alpha\vec{v}\|_2 = \|\alpha\| \cdot \|\vec{v}\|_2 \quad \forall \vec{v} \in \mathbb{R}^n, \forall \alpha \in \mathbb{R}$	***absolute homogeneity***
(9.9c)	$\|\vec{v} + \vec{w}\|_2 \leqq \|\vec{v}\|_2 + \|\vec{w}\|_2 \quad \forall\, \vec{v}, \vec{w} \in \mathbb{R}^n$	***triangle inequality***

Proof:
a. *It is certainly true that $\|\vec{v}\|_2 \geqq 0$ for any n–dimensional vector $\vec{v}$ because it is defined as $+\sqrt{K}$ where the quantity K is, as a sum of squares, non–negative. If 0 is the zero vector with coordinates $x_1 = x_2 = \ldots = x_n = 0$ then obviously $\|0\|_2 = \sqrt{0 + \ldots + 0} = 0$. Conversely, let $\vec{v} = (x_1, x_2, \dots, x_n)$ be a vector in $\mathbb{R}^n$ such that $\|\vec{v}\|_2 = 0$. This means that $\sqrt{\sum_{j=1}^{n} {x_j}^2} = 0$ which is only possible if everyone of the non–negative x_j is zero. In other words, $\vec{v}$ must be the zero vector 0.*

b. *Let $\vec{v} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then*

$$\begin{aligned} \|\alpha\vec{v}\|_2 &= \sqrt{\sum_{j=1}^{n} (\alpha x_j)^2} = \sqrt{\sum_{j=1}^{n} \alpha^2 \alpha {x_j}^2} = \sqrt{\alpha^2 \sum_{j=1}^{n} \alpha {x_j}^2} = \sqrt{\alpha^2}\sqrt{\sum_{j=1}^{n} \alpha {x_j}^2} \\ &= \sqrt{\alpha^2}\|\vec{v}\|_2 = |\alpha| \cdot \|\vec{v}\|_2 \end{aligned}$$

because it is true that $\sqrt{\alpha^2} = |\alpha|$ for any real number α (see assumption 2.1 on p.17).

c. *The proof will only be given for $n = 1, 2, 3$.*
***n* = 1** : *(9.9.c) simply is the triangle inequality for real numbers (see (2.2) on 17) and we are done.*
***n* = 2, 3** : *Look back at the picture about addition of vectors in the plane or in space (see p.153). Remember that for any two vectors $\vec{v}$ and $\vec{w}$ you can always build a triangle whose sides have length $\|\vec{v}\|_2$, $\|\vec{w}\|_2$ and $\|\vec{v} + \vec{w}\|_2$. It is clear that the length of any one side cannot exceed the sum of the lengths of the other two sides, so we get specifically $\|\vec{v} + \vec{w}\|_2 \leqq \|\vec{v}\|_2 + \|\vec{w}\|_2$ and we are done.*

The geometric argument is not exactly an exact proof but I used it nevertheless because it shows the origin of the term "triangle inequality" for property (9.9.c). An exact proof will be given for arbitrary $n \in \mathbb{N}$ as a consequence of the so–called Cauchy–Schwartz inequality (cor.9.1). The inequality itself is stated and proved in prop.9.9 on p.166 in the section which discusses inner products (dot products) on vector spaces. ■

9.2 General Vector Spaces

9.2.1 Vector spaces: Definition and Examples

Part of this follows [3] Brin, Matthew and Marchesi, Gerald: Linear Algebra, a text for Math 304, Spring 2016.

Mathematicians are very fond of looking at very different objects and figuring out what they have in common. They then create an abstract concept whose items have those properties and examine what they can conclude. For those of you who have had some exposure to object oriented programming: It's like defining a base class, e.g., "mammal", that possesses the core properties of several concrete items such as "horse", "pig", "whale" (sorry – can't require that all mammals have legs). We have looked at the following items that seem to be quite different:

real numbers
N–dimensional vectors
real functions

Well, that was disingenuous. I took great pains to explain that real numbers and one–dimensional vectors are sort of the same (see 9.3 on p.152). Besides I also explained that N–dimensional vectors can be thought of as real functions on the domain $X = \{1, 2, 3, \cdots, N\{$*. (see 9.4 on p.152). Never mind, I'll introduce you now to vector spaces as sets of objects which you can "add" and multiply with real numbers according to rules which are guided by those that apply to addition and multiplication of ordinary numbers.*

Here is quick reminder on how we add N–dimensional vectors and multiply them with scalars (real numbers) (see (9.1.2) on p.152). Given are two N–dimensional vectors
$\vec{x} = (x_1, x_2, \ldots, x_N)$ *and* $\vec{y} = (y_1, y_2, \ldots, y_N)$ *and a real number* α*. Then the sum* $\vec{z} = \vec{x} + \vec{y}$ *of* $\vec{x}$ *and* $\vec{y}$ *is the vector with the components*

$$z_1 = x_1 + y_1; \quad z_2 = x_2 + y_2; \quad \ldots; \quad z_N = x_N + y_N;$$

and the scalar product $\vec{w} = \alpha\vec{x}$ *of* α *and* $\vec{x}$ *is the vector with the components*

$$w_1 = \alpha x_1; \quad w_2 = \alpha x_2; \quad \ldots; \quad w_N = \alpha x_N;$$

Example 9.8 (Vector addition and scalar multiplication)**.** We use $N = 2$ in this example:
Let $a = (-3, 1/5),\ b = (5, \sqrt{2})$ We add those vectors by adding each of the coordinates separately:

$$a + b = (2, 1/5 + \sqrt{2})$$

and we multiply a with a scalar $\lambda \in \mathbb{R}$, e.g. $\lambda = 100$, by multiplying each coordinate with λ:

$$100a = 100(-3, 1/5) = (-300, 20). \ \square$$

In the last example I have avoided using the notation "$\vec{x}$*" with the cute little arrows on top for vectors. I did that on purpose because this notation is not all that popular in Math even for N–dimensional vectors and definitely not for the more abstract vectors as elements of a vector space. Here now is the definition of a vector space, taken almost word for word from the book "Introductory Real Analysis" (Kolmogorov/Fomin [6]). This definition is quite lengthy because a set needs to satisfy many rules to be a vector space.*

Definition 9.4 (Vector spaces (linear spaces))**.** A non–empty set V of elements $x, y, z, \dots$ is called a **vector space** or **linear space** if it satisfies the following:

A. Any two elements $x, y \in V$ uniquely determine a third element $x + y \in V$, called the **sum** of x and y with the following properties:

1. $x + y \;=\; y + x$ (**commutativity**);
2. $(x + y) + z \;=\; x + (y + z)$ (**associativity**);
3. There exists an element $0 \in V$, called the **zero element**, or **zero vector**, or **null vector**, with the property that $x + 0 = x$ for each $x \in V$;
4. For every $x \in V$, there exists an element $-x \in V$, called the **negative** of x, with the property that $x + (-x) = 0$ for each $x \in V$. When adding negatives, then there is a convenient short form. We write $x - y$ as an abbreviation for $x + (-y)$;

B. Any real number α and element $x \in V$ together uniquely determine an element $\alpha x \in V$ (sometimes also written $\alpha \cdot x$), called the **scalar product** of α and x. It has the following properties:

1. $\alpha(\beta x) \;=\; (\alpha\beta)x$;
2. $1x \;=\; x$;

C. The operations of addition and scalar multiplication obey the two **distributive laws**

1. $(\alpha + \beta)x \;=\; \alpha x + \beta x$;
2. $\alpha(x + y) \;=\; \alpha x + \alpha y$;

The elements of a vector space are called **vectors**. □

Definition 9.5 (Subspaces of vector spaces)**.** Let V be a vector space and let $A \subseteq V$ be a non–empty subset of V with the following property: For any $x, y \in A$ and $\alpha \in \mathbb{R}$ the sum $x + y$ and the scalar product αx also belong to A. Then A is called a **subspace** of V.

The set $\{0\}$ which contains the null vector 0 of V as its single element is called the **nullspace**. □

Remark 9.1 (Closure properties)**.**

- **a.** Note that if $\alpha = 0$ then $\alpha x = 0$. it follows that the null vector belongs to any subspace.
- **b.** We ruled out the case $A = \emptyset$ but did not require that A be a strict subset of V ((2.3) on p.11). In other words, the entire vector space V is a subspace of itself.
- **c.** It is trivial to verify that the nullspace $\{0\}$ is a subspace. □

Proposition 9.2 (Subspaces are vector spaces)**.** *A subspace of a vector space is a vector space, i.e., it satisfies all requirements of definition (9.4).*

Proof: None of the equalities that are part of the definition of a vector space magically ceases to be valid just because we look at a subset. The only thing that could go wrong is that some of the expressions might not belong to A anymore. I'll leave it to you to figure out why this won't be the case, but I'll show you the proof for the second distributive law of part C.

We must prove that for any $x, y \in A$ *and* $\lambda \in \mathbb{R}$

$$\lambda(x+y) \;=\; \lambda x + \lambda y :$$

First, $x + y \in A$ *because a subspace contains the sum of any two of its elements. It follows that* $\lambda(x+y)$ *as product of a real number with an element of A again belongs to A because it is a subspace. Hence the left hand side of the equation belongs to A.*

Second, both λx *and* λy *belong to A because each is the scalar product of* λ *with an element of A and this set is a subspace. It follows for the same reason that the right hand side of the equation as the sum of two elements of the subspace A belongs to A.*

Equality of $\lambda(x+y)$ *and* $\lambda x + \lambda y$ *is true because it is true if we look at* x *and* y *as elements of* V. ■

Remark 9.2 (Closure properties)**.** If a subset B of a larger set X has the property that certain operations on members of B will always yield elements of B, then we say that B is **closed** with respect to those operations. □

A subspace is a subset of a vector space which is closed with respect to vector addition and scalar multiplication.

You have already encountered the following examples of vector spaces:

Example 9.9 (Vector space $\mathbb{R}$)**.** The real numbers $\mathbb{R}$ are a vector space if you take the ordinary addition of numbers as "+" and the ordinary multiplication of numbers as scalar multiplication. □

Example 9.10 (Vector space $\mathbb{R}^n$)**.** More general, the sets $\mathbb{R}^n$ of n–dimensional vectors are vector spaces when you define addition and scalar multiplication as in (9.2) on p.152. To see why, just look at each component (coordinate) separately and you just deal with ordinary real numbers. □

The following remark should be thought of as the ***definition*** *of the very important function spaces* $\mathscr{F}(X,\mathbb{R})$, $\mathscr{B}(X,\mathbb{R})$, $\mathscr{C}(X,\mathbb{R})$.

Example 9.11 (Vector spaces of real–valued functions)**.** Let X be an arbitrary, non–empty set. Then

$$\mathscr{F}(X,\mathbb{R}) \;:=\; \{f(\cdot) : f(\cdot) \text{ is a real function on } X\} \tag{9.10}$$

denotes the set of all real functions with domain X [74] and

$$\mathscr{B}(X,\mathbb{R}) \;:=\; \{g(\cdot) : g(\cdot) \text{ is a bounded real function on } X\}$$

denotes the subset of all bounded real functions with domain X.

[74] Note that $\mathscr{F}(X,\mathbb{R}) = \mathbb{R}^X$ (see remark 5.4, p.101 which follows def.5.5 of the Cartesian Product of a family of sets.)

Let $A \subseteq \mathbb{R}$. Then

$$\mathscr{C}(A, \mathbb{R}) \ := \{\psi(\cdot) : \psi(\cdot) \text{ is a continuous real function on } A\}$$

denotes the set of all real–valued continuous functions with domain A. [75]

We list separately the case $X = [a, b]]$ where $a, b \in \mathbb{R}$ such that $a < b$. Then

$$\mathscr{C}([a, b], \mathbb{R}) \ := \ \{h(\cdot) : h(\cdot) \text{ is a continuous real function for } a \leqq x \leqq b\}$$

denotes the set of all contnuous real functions with domain $[a, b]$. Note that, for continuous functions, we had to restrict our choice of domain to subsets of real numbers because there is no notion of continuity for functions on abstract domains (and codomains).

If you define addition and scalar multiplication as in (4.13) on p.87, then each of these sets of real–valued functions becomes a vector space for the following two reasons:

I: You can verify properties A, B, C of a vector space by looking at the function values for a specific argument $x \in X$ because then you just deal with ordinary real numbers.

II: The sum of two bounded functions and the product of a bounded function with a scalar is a bounded function. In other words, "+" associates with any two elements $f, g \in \mathscr{B}(X, \mathbb{R})$ a third item $f + g \in \mathscr{B}(X, \mathbb{R})$ and "·" associates with any $f \in \mathscr{B}(X, \mathbb{R})$ and $\alpha \in \mathbb{R}$ a third item $\alpha \cdot f \in \mathscr{B}(X, \mathbb{R})$.

Likewise, the sum of two continuous functions and the product of a continuous function with a scalar is a continuous function. As for bounded functions, "+" associates with any two elements $f, g \in \mathscr{C}([a, b], \mathbb{R})$ a third item $f + g \in \mathscr{C}([a, b], \mathbb{R})$ and "·" associates with any $f \in \mathscr{C}([a, b], \mathbb{R})$ and $\alpha \in \mathbb{R}$ an item $\alpha \cdot f \in \mathscr{C}([a, b], \mathbb{R})$.

It follows from the above that all three function sets are vector spaces and also that **1)** $\mathscr{B}(X, \mathbb{R})$ is a subspace of $\mathscr{F}(X, \mathbb{R})$, **2)** $\mathscr{C}(X, \mathbb{R})$ is a subspace of $\mathscr{F}(X, \mathbb{R})$.

We will see in ch.**??** (Compactness) on p.**??** that continuous functions defined on a closed interval are bounded. It follows that

$$\mathscr{C}([a, b], \mathbb{R}) \ \subseteq \ \mathscr{B}([a, b], \mathbb{R}) \ \subseteq \ \mathscr{F}([a, b], \mathbb{R}).$$

We deduce from this that **3)** $\mathscr{C}([a, b], \mathbb{R})$ also is a subspace of $\mathscr{B}([a, b], \mathbb{R})$.

It should be noted though that, for example, continuous function need **not** be bounded on **open** intervals $]a, b[$, as the example $f(x) = \frac{1}{x}$ demonstrates for $a = 0$ and $b = 1$. □

Here are some more examples.

Example 9.12 (Subspace $\{(x, y) : x = y\}$ **).** The set $V := \{(x, x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$ of all vectors in the plane with equal x and y coordinates has the following property: For any two vectors $\vec{x} = (a, a)$ and $\vec{y} = (b, b) \in V$ $(a, b \in \mathbb{R})$ and real number α the sum $\vec{x} + \vec{y} = (a + b, a + b)$ and the scalar product $\alpha\vec{x} = (\alpha a, \alpha a)$ have equal x–and y–coordinates, i.e., they again belong to V. Moreover the zero–vector 0 with coordinates $(0, 0)$ belongs to V. It follows that the subset L of $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$ (see (9.5) on p.157). □

[75] Continuity for such functions was discussed in ch.8.2 on p.129.

A proof for the following is omitted even though it is not difficult:

Example 9.13 (Subspace $\{(x,y) : y = \alpha x\}$)**.** Any subset of the form

$$V_\alpha := \{(x,y) \in \mathbb{R}^2 : y = \alpha x\}$$

is a subspace of $\mathbb{R}^2$ ($\alpha \in \mathbb{R}$). Draw a picture: V_α is the straight line through the origin in the xy–plane with slope α. $\square$

Example 9.14 (Embedding of linear subspaces)**.** The last example was about the subspace of a bigger space. Now we switch to the opposite concept, the **embedding** of a smaller space into a bigger space. We can think of the real numbers $\mathbb{R}$ as a part of the xy–plane $\mathbb{R}^2$ or even 3–dimensional space $\mathbb{R}^3$ by identifying a number a with the two-dimensional vector $(a, 0)$ or the three-dimensional vector $(a, 0, 0)$. Let $M < N$. It is not a big step from here that the most natural way to uniquely associate an N–dimensional vector with an M–dimensional vector $\vec{x} := (x_1, x_2, \dots, x_M)$ by adding zero-coordinates to the right:

$$\vec{x} := (x_1, x_2, \dots, x_M, \underbrace{0, 0, \dots, 0}_{N-M \text{ times}}) \quad \square$$

Example 9.15 (All finite–dimensional vectors)**.** Let

$$\mathfrak{S} := \bigcup_{n \in \mathbb{N}} \mathbb{R}^n = \mathbb{R}^1 \cup \mathbb{R}^2 \cup \dots \cap \mathbb{R}^n \cup \dots$$

be the set of all vectors of finite (but unspecified) dimension.

We can define addition for any two elements $\vec{x}, \vec{y} \in \mathfrak{S}$ as follows: If $\vec{x}$ and $\vec{y}$ both happen to have the same dimension N then we add them as usual: the sum will be $x_1 + y_1, x_2 + y_2, \dots, x_N + y_N,$. If not, then one of them, say $\vec{x}$ will have dimension M smaller than the dimension N of $\vec{y}$. We now define the sum $\vec{x} + \vec{y}$ as the vector

$$\vec{z} := (x_1 + y_1, x_2 + y_2, \dots, x_M + y_M, y_{M+1}, y_{M+2}, \dots, y_N)$$

which is hopefully what you expected to happen. $\square$

Example 9.16 (All sequences of real numbers)**.** Let $\mathbb{R}^{\mathbb{N}} = \prod_{j \in \mathbb{N}} \mathbb{R}$ (see (5.5) on p.101). Is this the same set as $\mathfrak{S}$ from the previous example? The answer is No. Can you see why? I would be surprised if you do, so let me give you the answer: Each element $x \in \mathfrak{S}$ is of some finite dimension, say N, meaning that that it has no more than N coordinates. Each element $y \in \mathbb{R}^{\mathbb{N}}$ is a collection of numbers $y_1, y_2, \dots$ none of which need to be zero. In fact, $\mathbb{R}^{\mathbb{N}}$ is the vector space of all sequences of real numbers. Addition is of course done coordinate by coordinate and scalar multiplication with $\alpha \in \mathbb{R}$ is done by multiplying each coordinate with α.

There is again a natural way to embed $\mathfrak{S}$ into $\mathbb{R}^{\mathbb{N}}$ as follows: We transform an N–dimensional vector $(a_1, a_2, \dots, a_N)$ into an element of $\mathbb{R}^{\mathbb{N}}$ (a sequence $(a_j)_{j \in \mathbb{N}}$) by setting $a_j = 0$ for $j > N$. $\square$

Definition 9.6 (linear combinations)**.** Let V be a vector space and let $x_1, x_2, x_3, \dots, x_n \in V$ be a finite number of vectors in V. Let $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n \in \mathbb{R}$. We call the finite sum

$$\sum_{j=0}^{n} \alpha_j x_j = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \dots + \alpha_n x_n \tag{9.11}$$

a **linear combination** of the vectors x_j . The multipliers $\alpha_1, \alpha_2, \dots$ are called **scalars** in this context. □

In other words, linear combinations are sums of scalar multiples of vectors. You should understand that the expression in (9.11) always is an element of V, no matter how big $n \in \mathbb{N}$ was chosen:

Proposition 9.3 (Vector spaces are closed w.r.t. linear combinations)**.** *Let V be a vector space and let $x_1, x_2, x_3, \dots, x_n \in V$ be a finite number of vectors in V. Let $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n \in \mathbb{R}$. Then the linear combination $\sum\limits_{j=0}^{n} \alpha_j x_j$ also belongs to V. Note that this is also true for subspaces because those are vector spaces, too.*

Proof: Trivial. ■

Proposition 9.4. *Let V be a vector space and let $(W_i)_{i \in I}$ be a family of subspaces of V. Let $W := \bigcap \left[W_i : i \in I \right]$. Then W is a subspace of V.*

Proof: It suffices to show that W is not empty and that any linear combination of items in W belongs to W. As $0 \in W_i$ for each $i \in I$, it follows that $0 \in W$, hence $W \neq \emptyset$.

Let $x_1, x_2, \dots x_k \in W$ and $\alpha_1, \alpha_2, \dots \alpha_k \in \mathbb{R}(k \in \mathbb{N})$. Let $x := \sum_{j=1}^{k} \alpha_j x_j$. Then $x \in W_i$ for all i because each W_i is a vector space, hence $x \in W$. ■

Definition 9.7 (Linear span)**.** Let V be a vector space and $A \subseteq V$. Then $span(A)$:= the set of all linear combinations of vectors in A is called the **span** or **linear span** of A. In other words,

$$span(A) = \{ \sum_{j=1}^{k} \alpha_j x_j : k \in \mathbb{N}, \alpha_j \in \mathbb{R}, x_j \in A \; (1 \leqq j \leqq k) \}. \quad \square \tag{9.12}$$

Proposition 9.5. *Let V be a vector space and $A \subseteq V$. Then $span(A)$ is a subspace of V.*

Proof: Let $y_j \in span(A)$ for $j = 1, 2, \dots k$, i.e. y_j is a linear combination of vectors $x_{j,1}, x_{j,2}, \dots x_{j,n_j} \in A$. But then any linear combination of $y_1, y_2, \dots y_k$ is a linear combination of the vectors

$(x_{1,1}, x_{1,2}, \dots x_{1,n_1}),\ (x_{2,1}, x_{2,2}, \dots x_{2,n_2}), \dots, (x_{k,1}, x_{k,2}, \dots x_{k,n_k})$. ■

Theorem 9.1. *Let V be a vector space and $A \subseteq V$. Let $\mathfrak{V} := \{W \subseteq V : W \supseteq A \text{ and } W \text{ is a subspace of } V\}$. Then $span(A) = \bigcap \left[W : W \in \mathfrak{V} \right]$.*

Clearly, $span(A) \supseteq A$ It follows from prop.9.5 that $span(A) \in \mathfrak{V}$, hence $span(A) \supseteq \bigcap \left[W : W \in \mathfrak{V} \right]$.

On the other hand, Any subspace W of V that contains A also contains all its linear combinations, hence $span(A) \subseteq W$ for all $W \in \mathfrak{V}$. But then $span(A) \subseteq \bigcap \left[W : W \in \mathfrak{V} \right]$. ■

Remark 9.3 (Linear span(A) = subspace generated by A)**.** Let V be a vector space and $A \subseteq V$. Theorem 9.1 justifies to call $span(A)$:= the **subspace generated by** A. □

Definition 9.8 (linear mappings)**.** Let V_1, V_2 be two vector spaces.

Let $f(\cdot) : V_1 \to V_2$ be a mapping with the following properties:

(9.13a) $f(x+y) \;=\; f(x)+f(y) \quad \forall x,y \in V_1$ **additivity**

(9.13b) $f(\alpha x) \;=\; \alpha f(x) \quad \forall x \in V_1,\ \forall \alpha \in \mathbb{R}$ **homogeneity**

Then we call $f(\cdot)$ a **linear mapping**. □

Note 9.1 (Note on homogeneity)**.** We encountered homogeneity when looking at the properties of the Euclidean norm ((9.9) on p.155), but homogeneity is defined differently there in that you had to take the absolute value $|\alpha|$ instead of α. □

Remark 9.4 (Linear mappings are compatible with linear combinations)**.** We saw in the last proposition that vector spaces are closed with respect to linear combinations. Linear mappings and linear combinations go together very well in the following sense:

Remember that for any kind of mapping $x \mapsto f(x)$, $f(x)$ was called the image of x. Now we can express what linear mappings are about like this:

A: The image of the sum is the sum of the image

B: The image of the scalar multiple is the scalar multiple of the image

C: The image of the linear combination is the linear combination of the images

Mathematicians express this by saying that linear mappings **preserve** or are **compatible with** linear combinations. □

Proposition 9.6 (Linear mappings preserve linear combinations)**.** *Let V_1, V_2 be two vector spaces.*

Let $f(\cdot) : V_1 \to V_2$ be a linear map and let $x_1, x_2, x_3, \dots, x_n \in V_1$ be a finite number of vectors in the domain V_1 of $f(\cdot)$. Let $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n \in \mathbb{R}$. Then $f(\cdot)$ preserves any such linear combination:

(9.14)
$$f\Big(\sum_{j=0}^{n} \lambda_j x_j\Big) \;=\; \sum_{j=0}^{n} \lambda_j f(x_j).$$

Proof:
First we note that $f(\lambda_j x_j) \;=\; \lambda_j f(x_j)$ for all j because linear mappings preserve scalar multiples and the proof is done for $n=1$. Because they also preserve the addition of any two elements, the proposition holds for $n=2$. We prove the general case by induction (see (2.12) on p.18). Our induction assumption is

$$f\Big(\sum_{j=0}^{n-1} \lambda_j x_j\Big) \;=\; \sum_{j=0}^{n-1} \lambda_j f(x_j).$$

We use it in the third equality of the following:

$$f\Big(\sum_{j=0}^{n} \lambda_j x_j\Big) \;=\; f\Big(\sum_{j=0}^{n-1} \lambda_j x_j + \lambda_n x_n\Big) = f\Big(\sum_{j=0}^{n-1} \lambda_j x_j\Big) + f(\lambda_n x_n) = \sum_{j=0}^{n-1} \lambda_j f(x_j) + f(\lambda_n x_n) = \sum_{j=0}^{n} \lambda_j f(x_j)$$

■

Here are some examples of linear mappings.

Example 9.17 (Projection on the first coordinate)**.** Let $N \in \mathbb{N}$. The map

$$\pi_1(\cdot) : \mathbb{R}^N \to \mathbb{R} \qquad (x_1, x_2, \dots, x_N) \mapsto x_1$$

is called the **projection** on the first coordinate or the first coordinate function.

Example 9.18 (Projections on any coordinate)**.** More generally, let $N \in \mathbb{N}$ and $1 \leqq j \leqq N$. The map

$$\pi_j(\cdot) : \mathbb{R}^N \to \mathbb{R} \qquad (x_1, x_2, \dots, x_N) \mapsto x_j$$

is called the **projection** on the jth coordinate or the j**th coordinate function**.

It is easy to see what that means if you set $N = 2$: For the two–dimensional vector $\vec{v} := (3.5, -2) \in \mathbb{R}^2$ you get $\pi_1(\vec{v}) = 3.5$ and $\pi_2(\vec{v}) = -2$. □

Example 9.19 (Projections on any lower dimensional space)**.** In the last two examples we projected $\mathbb{R}^N$ onto a one–dimensional space. More generally, we can project $\mathbb{R}^N$ onto a vector space $\mathbb{R}^M$ of lower dimension M (i.e., we assume $M < N$) by keeping M of the coordinates and throwing away the remaining $N - M$. Mathematicians express this as follows:

Let $M, N, i_1, i_2, \dots, i_M \in \mathbb{N}$ such that $M < N$ and $1 \leqq i_1 < i_2 < \dots < i_M \leqq N$. The map

(9.15) $$\pi_{i_1, i_2, \dots, i_M}(\cdot) : \mathbb{R}^N \to \mathbb{R}^M \qquad (x_1, x_2, \dots, x_N) \mapsto (x_{i_1}, x_{i_2}, \dots, x_{i_M})$$

is called the **projection** on the coordinates $i_1, i_2, \dots, i_M$. [76] □

Example 9.20. Let $x_0 \in A$. The mapping

(9.16) $$\varepsilon_{x_0} : \mathscr{F}(A, \mathbb{R}) \to \mathbb{R}; \qquad f(\cdot) \mapsto f(x_0)$$

which assigns to any real function on A its value at the specific point x_0 is a linear mapping because if $h(\cdot) = \sum\limits_{j=0}^{n} a_j f_j(\cdot)$ then

$$\varepsilon_{x_0}\Big(\sum_{j=0}^{n} a_j f_j(\cdot)\Big) = \varepsilon_{x_0}(h(\cdot)) = h(x_0) = \sum_{j=0}^{n} a_j f_j(x_0) = \sum_{j=0}^{n} a_j \varepsilon_{x_0}(f_j(\cdot))$$

and this proves the linearity of the mapping $\varepsilon_{x_0}(\cdot)$. The mapping $\varepsilon_{x_0}(\cdot)$ is called the **abstract integral** with respect to point mass at x_0. □

Lemma 9.1 ($F \circ span = span \circ F$)**.** *[3] Brin/Marchesi Linear Algebra, general lemma 4.1.7: Let V, W be two vector spaces and $F : V \to W$ a linear mapping from V to W. Let $A \subseteq V$. Then*

(9.17) $$F(span(A)) = span(F(A)).$$

Proof: *See Brin/Marchesi Linear Algebra, general lemma 4.1.7.* ■

[76] You previously encountered an example where we made use of the projection

$$\pi_{1,2}(\cdot) : \mathbb{R}^3 \to \mathbb{R}^2 \qquad (x, y, z) \mapsto (x, y).$$

This was in the course of computing the length of a 3–dimensional vector (see (9.5) on p.152).

Definition 9.9 (Linear dependence and independence)**.** Let V be a vector space and $A \subseteq V$

a. A is called **linearly dependent** if the following is true: There exist distinct vectors $x_1, x_2, \dots x_k \in A$ and scalars $\alpha_1, \alpha_2, \dots \alpha_k \in \mathbb{R}$ ($k \in \mathbb{N}$) such that not all scalars α_j are zero ($1 \leqq j \leqq k$) and $\sum_{j=1}^{k} \alpha_j x_j \ = \ 0.$

b. A is called **linearly independent** if A is not linearly dependent, i.e., if the following is true: Let $x_1, x_2, \dots x_k \in A$ and $\alpha_1, \alpha_2, \dots \alpha_k \in \mathbb{R}$ ($k \in \mathbb{N}$). If $\sum_{j=1}^{k} \alpha_j x_j \ = \ 0$ then $\alpha_j = 0$ for all $1 \leqq j \leqq k$. □

Definition 9.10 (Basis of a vector space)**.** Let V be a vector space and $B \subseteq V$. B is called a **basis** of V if **a.** B is linearly independent and **b.** $span(B) = V$. □

Lemma 9.2. *Let V be a vector space and $A \subseteq V$ linearly independent. If $span(A) \subsetneq V$ and $y \in span(A)^{\complement}$ then $A' := A \cup \{y\}$ is linearly independent.*

Proof:
Let $x_1, x_2, \dots x_k \in A'$ and $\alpha_1, \alpha_2, \dots \alpha_k \in \mathbb{R}$ ($k \in \mathbb{N}$) such that

$$\sum_{j=1}^{k} \alpha_j x_j \ = \ 0 \tag{9.18}$$

We must show that each α_j is zero.

***Case 1**: $y \neq x_j$ for all j:*
Then $y \in A$ and it follows from the linear independence of A that each α_j is zero.

***Case 2**: $y = x_{j_0}$ for some $1 \leq j_0 \leqq k$:*
We first show that $\alpha_{j_0} = 0$: Otherwise

$$x_{j_0} \ = \ \sum_{j \neq j_0} \frac{-\alpha_j}{\alpha_{j_0}} x_j \tag{9.19}$$

is a linear combination of elements of A, contrary to the assumption that $x_{j_0} = y \in span(A)^{\complement}$ and we have shown that $\alpha_{j_0} = 0$.

It follows from (9.18) that

$$\sum_{j \neq j_0} \alpha_j x_j \ = \ 0 \tag{9.20}$$

and It follows as in case 1 from the linear independence of A that if $j \neq j_0$ then α_j also is zero. ■

9.2.2 Normed Vector Spaces (Study this!)

Definition 9.3 on p.154 in ch.9.1.3 (Length of N–Dimensional Vectors and the Euclidean Norm) gave the definition of the Euclidean norm $\|\vec{x}\|_2 = \sum_{j=1}^{N} x_j^2$ in $\mathbb{R}_N$. We saw that in dimensions $n = 1, 2, 3$ that $\|\vec{x}\|_2$

equals the length of the vector $\vec{x}$ and that prop.9.1 on p. 155 "proved" informally for $n = 1, 2, 3$ that $\| \cdot \|_2$ satisfies the following three properties:

- ***a.*** *positive definiteness,*
- ***b.*** *absolute homogeneity,*
- ***c.*** *triangle inequality.*

In this chapter we define the norm $\|x\|$ of a vector x in an abstract vector space as a function which satisfies the above three properties, and hence generalizes the concept of the length of a vector in N–dimensional space to more general vector spaces. Before we give give this definition, we first introduce the concept of an inner product $x \bullet y$ of two vectors x and y. We will see that some of the most important norms, the Euclidean norm among them, can be derived from inner products.

The following definition of inner products and proof of the Cauchy–Schwartz inequality were taken from "Calculus of Vector Functions" (Williamson/Crowell/Trotter [10]).

Definition 9.11 (Inner products)**.** Let V be a vector space with a function

$$\bullet(\cdot,\cdot) : V \times V \to \mathbb{R}; \qquad (x, y) \mapsto x \bullet y := \bullet(x, y)$$

which satisfies the following properties:

(9.21a)	$x \bullet x \geqq 0 \quad \forall x \in V$ and $x \bullet x = 0 \;\Leftrightarrow\; x = 0$	**positive definiteness**
(9.21b)	$x \bullet y = y \bullet x \quad \forall x, y \in V$	**symmetry**
(9.21c)	$(x + y) \bullet z = x \bullet z + y \bullet z \quad \forall\, x, y, z \in V$	**additivity**
(9.21d)	$(\lambda x) \bullet y = \lambda(x \bullet y) \quad \forall\, x, y \in V \quad \forall\, \lambda \in \mathbb{R}$	**homogeneity**

We call such a function an **inner product**. [77] $\square$

Note that additivity and homogeneity of the mapping $x \mapsto x \bullet y$ for a fixed $y \in V$ imply linearity of that mapping and the symmetry property implies that the mapping $y \mapsto x \bullet y$ for a fixed $x \in V$ is linear too. In other words, an inner product is binear in the following sense:

Definition 9.12 (Bilinearity)**.** Let V be a vector space with a function

$$F(\cdot,\cdot) : V \times V \to \mathbb{R}; \qquad (x, y) \mapsto F(x, y).$$

$F(\cdot,\cdot)$ is called **bilinear** if it is linear in each component, i.e., the mappings

$$F_1 : V \to \mathbb{R}; \quad x \mapsto F(x, y)$$
$$F_2 : V \to \mathbb{R}; \quad y \mapsto F(x, y)$$

are both linear. $\square$

Proposition 9.7 (Algebraic properties of the inner product)**.** *Let V be a vector space with inner product $\bullet(\cdot,\cdot)$. Let $a, b, x, y \in V$. Then*

(9.22a) $\quad (a + b) \bullet (x + y) = a \bullet x \;+\; b \bullet x \;+\; a \bullet y \;+\; b \bullet y$

(9.22b) $\quad (x + y) \bullet (x + y) = x \bullet x \;+\; 2(x \bullet y) \;+\; y \bullet y$

(9.22c) $\quad (x - y) \bullet (x - y) = x \bullet x \;-\; 2(x \bullet y) \;+\; y \bullet y$

[77] also called **dot product**, e.g., in [3] Brin/Marchesi Linear Algebra, ch.6, Orthogonality.

*Proof of **a:***

$$(a+b)\bullet(x+y) = (a+b)\bullet x \ + \ (a+b)\bullet y$$
$$= a\bullet x \ + \ b\bullet x \ + a\bullet y \ + \ b\bullet y.$$

We used linearity in the second argument for the first equality and linearity in the first argument for the second equality.

*Proof of **b** and **c**: Left as an exercise.*

The following is the most important example of an inner product.

Proposition 9.8 (Inner product on $\mathbb{R}^N$)). *Let $N \in \mathbb{N}$. Then the real–valued function*

(9.23) $$(\vec{v},\vec{w}) \ \mapsto \ x_1y_1 + x_2y_2 + \ldots + x_Ny_N \ = \ \sum_{j=1}^{N} x_jy_j$$

is an inner product on $\mathbb{R}^N \times \mathbb{R}^N$.

Proof:

- ***a.*** *For $\vec{v} = \vec{w}$ we obtain $\vec{v}\bullet\vec{v} \ = \ \sum_{j=1}^{N} x_j^2$ and positive definiteness of the inner product follows from* $\sum_{j=1}^{N} x_j^2 = 0 \ \Leftrightarrow \ x_j^2 = 0 \ \forall j \ \Leftrightarrow \ x_j = 0 \ \forall j$
- ***b.*** *Symmetry is clear because $x_jy_j = y_jx_j$.*
- ***c.*** *Additivity follows from the fact that $(x_j+y_j)z_j = x_jz_j + y_jz_j$.*
- ***d.*** *Homogeneity follows from the fact that $(\lambda x_j)y_j = \lambda(x_jy_j)$.* ■

Proposition 9.9 (Cauchy–Schwartz inequality for inner products). *Let V be a vector space with an inner product*

$$\bullet(\cdot,\cdot) : V\times V \to \mathbb{R}; \qquad (x,y)\mapsto x\bullet y := \bullet(x,y)$$

Then

$$(x\bullet y)^2 \ \leqq \ (x\bullet x)\ (y\bullet y).$$

Proof:
Step 1: *We assume first that $x\bullet x = y\bullet y = 1$. Then*

$$0 \ \leqq (x-y\bullet x-y)$$
$$= x\bullet x - 2x\bullet y + y\bullet y = 2 - 2(x\bullet y)$$

where the first equality follows from proposition (9.7) on p.165.
This means $2(x\bullet y) \ \leqq \ 2$, i.e., $x\bullet y \ \leqq \ 1 \ = \ (x\bullet x)\ (y\bullet y)$ where the last equality is true because we had assumed $x\bullet x = y\bullet y = 1$. The Cauchy–Schwartz inequality is thus true under that special assumption.

Step 2: *General case: We do not assume anymore that $x\bullet x = y\bullet y = 1$. If x or y is zero then the Cauchy–Schwartz inequality is trivially true because, say if $x = 0$ then the left hand side becomes*

$$(x\bullet y)^2 = (0x\bullet y)^2 = 0(x\bullet y)^2 = 0$$

whereas the right hand side is, as the product of two non–negative numbers $x \bullet x$ *and* $y \bullet y$ *, non–negative.*

So we can assume that x *and* y *are not zero. On account of the positive definiteness we have* $x \bullet x > 0$ *and* $y \bullet y > 0$. *This allows us to define* $u := x/\sqrt{x \bullet x}$ *and* $v := y/\sqrt{y \bullet y}$. *But then*

$$\begin{aligned} u \bullet u &= (x \bullet x)/\sqrt{x \bullet x}^2 &= 1 \\ v \bullet v &= (y \bullet y)/\sqrt{y \bullet y}^2 &= 1. \end{aligned}$$

We have already seen in step 1 that $u \bullet v \leqq 1$. *It follows that*

$$(x \bullet y)/(\sqrt{x \bullet x}\sqrt{y \bullet y}) \;=\; (x/\sqrt{x \bullet x}) \bullet (y/\sqrt{y \bullet y}) \;\leqq\; 1$$

We multiply both sides with $\sqrt{x \bullet x}\sqrt{y \bullet y}$*,*

$$x \bullet y \;\leqq\; \sqrt{x \bullet x}\sqrt{y \bullet y}.$$

We replace x *by* $-x$ *and obtain*

$$-(x \bullet y) \;\leqq\; \sqrt{x \bullet x}\sqrt{y \bullet y}.$$

Think for a moment about the meaning of the absolute value and it is clear that the last two inequalities together prove that

$$|x \bullet y| \;\leqq\; \sqrt{x \bullet x}\sqrt{y \bullet y}.$$

We square this and obtain

$$(x \bullet y)^2 \;\leqq\; (x \bullet x)\,(y \bullet y)$$

and the Cauchy–Schwartz inequality is proved. ■

Definition 9.13 (sup–norm of bounded real functions)**.** Let X be an arbitrary, non-empty set. Let $f : X \to \mathbb{R}$ be a bounded real function on X, i.e., there exists a (possibly very large) number K such that $|f(x)| \leqq K$ for all $x \in X$. [78] Let

$$\|f\|_\infty := \sup\{|f(x)| : x \in X\} \tag{9.24}$$

We call $\|f\|_\infty$ the **supremum norm** or **sup–norm** of the function f. □

Proposition 9.10 (Properties of the sup norm)**.** *Let* X *be an arbitrary, non–empty set. Let*

$$\mathscr{B}(X, \mathbb{R}) := \{h(\cdot) : h(\cdot) \text{ is a bounded real function on } X\}$$

(see example 9.11 on p. 158). Then the function

$$\|\cdot\|_\infty : \mathscr{B}(X, \mathbb{R}) \to \mathbb{R}_+, \qquad h \mapsto \|h\|_\infty = \sup\{|h(x)| : x \in X\}$$

which assigns to a bounded function on X *its sup–norm satisfies the following:*

(9.25a)	$\|f\|_\infty \geqq 0 \;\forall\, f \in \mathscr{B}(X, \mathbb{R})$ *and* $\|f\|_\infty = 0 \;\Leftrightarrow\; f(\cdot) = 0$	***positive definiteness***
(9.25b)	$\|\alpha f(\cdot)\|_\infty = \|\alpha\| \cdot \|f(\cdot)\|_\infty \;\forall\, f \in \mathscr{B}(X, \mathbb{R}), \forall\, \alpha \in \mathbb{R}$	***absolute homogeneity***
(9.25c)	$\|f(\cdot) + g(\cdot)\|_\infty \leqq \|f(\cdot)\|_\infty + \|g(\cdot)\|_\infty \;\forall\, f, g \in \mathscr{B}(X, \mathbb{R})$	***triangle inequality***

[78] see def.8.6 (bounded functions) on p.127

The proof is left as exercise 9.1.

Note 9.2. We previously discussed the Euclidean norm

$$\|\vec{x}\|_2 \ = \ \sqrt{\sum_{j=1}^{n} {x_j}^2} \tag{9.26}$$

for n–dimensional vectors $\vec{x} = (x_1, x_2, \dots, x_n)$. You saw in (9.1) on p.155 that it satisfies positive definiteness, absolute homogeneity and the triangle inequality, just like the sup–norm. [79] Those are properties which you associate with the length or size of an object. A very rich mathematical theory can be developed for a generalized definition of length which is based just on those properties. ☐

As mentioned before, mathematicians like to define new objects that are characterized by a certain set of properties. As an example we had the definition of a vector space which encompasses objects as different as finite–dimensional vectors and real functions. Accordingly we give a special name to a function defined on a vector space which satisfies positive definiteness, homogeneity and the triangle inequality.

Definition 9.14 (Normed vector spaces)**.** Let V be a vector space. A **norm** on V is a real function

$$\|\cdot\|: \ V \to \mathbb{R} \qquad x \mapsto \|x\|$$

with the following three properties:

(9.27a)	$\|x\| \geqq 0 \quad \forall x \in V \quad$ and $\quad \|x\| = 0 \ \Leftrightarrow \ x = 0$	**positive definiteness**
(9.27b)	$\|\alpha x\| = \|\alpha\| \cdot \|x\| \quad \forall x \in V, \forall \alpha \in \mathbb{R}$	**absolute homogeneity**
(9.27c)	$\|x + y\| \leqq \|x\| + \|y\| \quad \forall\, x, y \in V$	**triangle inequality**

We call V a **normed vector space** and we write $(V, \|\cdot\|)$ instead of V when we wish to emphasize what norm on V we are dealing with. ☐

Proposition 9.11. *Let $(V, \|\cdot\|)$ be a normed vector space and let $\gamma > 0$. Let $p: \ V \to \mathbb{R}$ be defined as $p(x) := \gamma\|x\|$. Then p also is a norm.*

Proof: Left as an exercise. ■

Definition 9.15 (p–norms for $\mathbb{R}^n$)**.** Let $p \geqq 1$. Then

$$\vec{x} \mapsto \|\vec{x}\|_p \ := \Big(\sum_{j=1}^{n} |x_j|^p\Big)^{1/p} \tag{9.28}$$

is a norm on $\mathbb{R}^n$). This norm is called the p**-norm** .

The Euclidean norm is a p–norm; it is the 2-norm. ☐

[79] Actually, the proof that $\|\cdot\|_2$ satisfies the triangle inequality was given only for dimensions $1, 2, 3$. It will be proved in this chapter that it is true for all dimensions n. See cor.9.1 (Inner products define norms) on p.169.

Remark 9.5. We have seen that a vector space can be endowed with more than one norm.

- **a.** Prop.9.11 which proved that if $x \mapsto \|x\|$ is a norm on a vector space V and $\beta > 0$ then $x \mapsto \beta \cdot \|x\|$ also is a norm on V.
- **b.** The p–norms for $\mathbb{R}^n$. $\square$

The following theorem shows that an inner product can be associated in a natural fashion with a norm.

Theorem 9.2 (Inner products define norms). *Let V be a vector space with an inner product*

$$\bullet(\cdot,\cdot) : V \times V \to \mathbb{R}; \qquad (x,y) \mapsto x \bullet y$$

Then

$$\|\cdot\|_\bullet : x \ \mapsto \ \|x\| \ = \ \sqrt{(x \bullet x)}$$

defines a norm on V

Proof:
Positive definiteness : *follows immediately from that of the inner product.*

Absolute homogeneity : *Let $x \in V$ and $\lambda \in \mathbb{R}$. Then*

$$\|\lambda x\|_\bullet \ = \sqrt{(\lambda x) \bullet (\lambda x)} \ = \ \sqrt{\lambda\lambda(x \bullet x))} \ = \ |\lambda|\sqrt{x \bullet x} \ = \ |\lambda|\|x\|_\bullet.$$

Triangle inequality : *Let $x, y \in V$. Then*

$$\begin{aligned}
\|x+y\|_\bullet{}^2 \ &= (x+y) \bullet (x+y) \\
&= x \bullet x \ + \ 2(x \bullet y) \ + \ y \bullet y \\
&\leqq x \bullet x \ + \ 2|x \bullet y| \ + \ y \bullet y \\
&\leqq x \bullet x \ + \ 2\sqrt{x \bullet x}\sqrt{y \bullet y} \ + \ y \bullet y \\
&= \|x\|_\bullet{}^2 \ + \ 2\|x\|_\bullet \, \|y\|_\bullet \ + \ \|y\|_\bullet{}^2 \\
&= \big(\|x\|_\bullet \ + \ \|y\|_\bullet\big)^2.
\end{aligned}$$

The second equation uses bilinearity and symmetry of the inner product. The first inequality expresses the simple fact that $\alpha \leqq |\alpha|$ for any number α. The second inequality uses Cauchy–Schwartz. The next equality just substitutes the definition $\|x\|_\bullet = \sqrt{(x \bullet x)}$ of the norm. The next and last equality is the binomial expansion $(a+b)^2 = a^2 + 2ab + b^2$ for the ordinary real numbers $a = \|x\|_\bullet$ and $b = \|y\|_\bullet$.

We take square roots in the above inequality $\|x+y\|_\bullet{}^2 \ \leqq \ \big(\|x\|_\bullet \ + \ \|y\|_\bullet\big)^2$ and obtain $\|x+y\|_\bullet \ \leqq \ \|x\|_\bullet \ + \ \|y\|_\bullet$, the triangle inequality we set out to prove. $\blacksquare$

It was stated in prop.9.1 on p. 155 that the Euclidean norm is in fact a norm but only positive definiteness and homogeneity were proved. We now can easily complete the proof.

Corollary 9.1. *The Euclidean norm in $\mathbb{R}^n$ defined as $\|(x_1, x_2, \dots, x_n)\|_2 \ = \ \sqrt{\sum\limits_{j=1}^{n} {x_j}^2}$ (see def. 9.3 on p.154) is a norm.*

Proof: This follows from the fact that

$$\vec{x} \bullet \vec{y} = \sum_{j=1}^{n} x_j y_j \qquad \text{where } \vec{x} = (x_1, \dots, x_n) \text{ and } \vec{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$$

defines an inner product on $\mathbb{R}^n \times \mathbb{R}^n$ (see prop.9.8) for which $\|(x_1, x_2, \dots, x_n)\|_2$ is the associated norm. ■

Definition 9.16 (Norm for an inner product)**.** Let V be a vector space with an inner product

$$\bullet(\cdot,\cdot) : V \times V \to \mathbb{R}; \qquad (x, y) \mapsto x \bullet y$$

Then

$$\|\cdot\|_\bullet : x \mapsto \|x\|_\bullet := \sqrt{(x \bullet x)} \tag{9.29}$$

is called the **norm associated with the inner product** $\bullet(\cdot,\cdot)$. □

We now look at an inner product on the vector space $\mathscr{C}([a,b],\mathbb{R})$ of all continuous real–valued functions on the interval $[a,b]$ which was defined in example 9.11 (Vector spaces of real functions) on p.158. We use the terminology of [8] Stewart, J: Single Variable Calculus) for the following.

Definition 9.17. Let $a, b \in \mathbb{R},\ a < b$ and assume that $f, g : [a,b] \to \mathbb{R}$ are integrable (example 4.19 on p.80) functions.

a. We call the definite integral $\int_a^b f(x)dx$ the **net area** between the graph of f, the x–axis, and the vertical lines through $(a,0)$ $(y = a)$ and $(b,0)$ $(y = b)$. The above integral treats areas above the x–axis as positive and below the x–axis as negative, i.e., the net area is the difference between the areas above the x–axis and those below the x–axis.

b. We call $\int_a^b |f(x)|dx$ the **area** between the graph of f, the x–axis, and the vertical lines $y = a$ and $y = b$. Note that $f(x)$ has been replaced by its absolute value $|f(x)|$. In contrast to the net area, areas below the x–axis are also counted positive. □

c. We call $\int_a^b f(x) - g(x)dx$ the **net area** between the graphs of f and g and the vertical lines $y = a$ and $y = b$. We call $\int_a^b |f(x) - g(x)|dx$ the **area** between the graphs of f and g and the vertical lines $y = a$ and $y = b$. □

Example 9.21. Let $f : [-1,1];\quad x \mapsto 4x^3$. The antiderivative (see example 4.19 on p.80) of f. is x^4 and we compute net area and area as follows:

a. Net area = $\int_{(-1)}^{1} 4x^3 dx = x^4\Big|_{-1}^{1} = 1 - 1 = 0;$

b. Area = $\int_{(-1)}^{1} 4|x^3| dx = -x^4\Big|_{-1}^{0} + -x^4\Big|_{0}^{1} = \big(0 - (-1)\big) + \big(1 - 0\big) = 2.$ □

Let $a, b \in \mathbb{R}$ such that $a < b$. We recall from example 10.9 on p.215 that $\mathscr{C}([a,b],\mathbb{R})$ denotes the vector space of all continuous real–valued functions on the interval $[a,b]$. We further remember from example 4.19 on *p.80) that continuous functions are integrable. This allows us to compare for* $f \in \mathscr{C}([a,b],\mathbb{R})$ *the expressions*

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}, \quad \int_a^b |f(x)|dx, \quad \text{and} \quad \int_a^b \big(f(x)\big)^2 dx. \tag{9.30}$$

All three expressions give in a sense the size of f*. The sup–norm measures it as the biggest possible displacement from zero, the integral over the absolute value measures the area between the gaps of the functions* $x \mapsto f(x)$ *and* $x \mapsto 0$*, and the last expression does the same with the square of* f*. In many respects the use of areas is considered superior to using the biggest difference to zero.*

Squaring f *rather than using its absolute value has some mathematical advantages. One of them is that this will define an inner product on* $\mathscr{C}([a,b],\mathbb{R})$*. We will discuss that now. In preparation we prove the following proposition.*

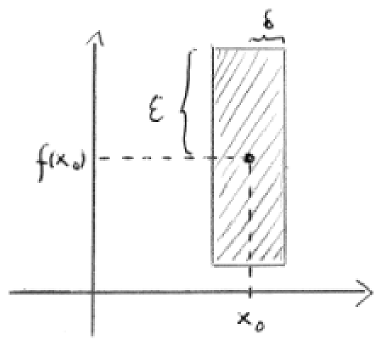
Proposition 9.12. *Let* $a,b \in \mathbb{R}$ *such that* $a < b$*. and let* $f : [a,b] \to [0,\infty[$ *be continuous. Then* $\int_a^b f(x)dx = 0$ *only if* $f(x) = 0$ *for all* $x \in \,]a,b[$*.* ☐

Proof: Assume that there is $a < x_0 < b$ *such that* $f(x_0) \neq 0$*, i.e.,* $f(x_0) > 0$*. Let* $\varepsilon := \frac{f(x_0)}{2}$*. As* f *is continuous at* x_0 *there exists according to thm.8.1 on p.136 some* $\delta > 0$ *such that*

$$|f(x_0) - f(x)| < \varepsilon, \text{ hence } f(x) > f(x_0) - \varepsilon = \varepsilon \text{ for all } x_0 - \delta < x < x_0 + \delta. \tag{9.31}$$

Continuity at x_0*:*

If $|x - x_0| < \delta$ *then* $|f(x_0) - f(x)| < \varepsilon$*: The graph of* f *stays within the rectangle with corners* $(x_0 \pm \delta, f(x_0) \pm \varepsilon)$*.*



Let $g : [a,b] \to \mathbb{R}$ *be defined as follows.*

$$g(x,y) = \begin{cases} \varepsilon & \text{if } x_0 - \delta < x < x_0 + \delta \\ 0 & \text{else.} \end{cases}$$

It follows from (9.31) that $f \geqq g$*, hence* $\int_a^b f(x)dx \geqq \int_a^b g(x)dx = (2\delta)\varepsilon > 0.$ ■

Proposition 9.13. *Let* $a,b \in \mathbb{R}$ *such that* $a < b$*. Then the mapping*

$$(f,g) \mapsto f \bullet g := \int_a^b f(x)g(x)dx \tag{9.32}$$

defines an inner product on $f \in \mathscr{C}([a,b],\mathbb{R})$ *.* ☐

Proof: We must prove positive definiteness, symmetry, and linearity in the left argument. In the following let $f,g,h \in \mathscr{C}([a,b],\mathbb{R})$ *and* $\lambda \in \mathbb{R}$*.*

a. *Positive definiteness: It follows from* $f^2(x) \geqq 0$ *that* $f \bullet f = \int_a^b f^2(x)dx \geqq 0$*. Clearly, if* 0 *denotes as usual the zero function* $x \mapsto 0$ *then* $0 \bullet 0 = 0$*. It remains to be shown that if* $\int_a^b f^2(x)dx \geqq 0$ *then* $f = 0$*. This follows from prop.9.12.*

b. *Symmetry:*

$$f \bullet g = \int_a^b f(x)g(x)dx = \int_a^b g(x)f(x)dx = g \bullet f.$$

c. *Additivity and homogeneity: This can be deduced from the well–known formulas*

$$\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx \quad \text{and} \quad \int_a^b \lambda g(x)dx = \lambda \int_a^b g(x)dx.$$

as follows:

(9.33)
$$(f+g) \bullet h = \int_a^b \big(f(x) + g(x)\big)h(x)dx = \int_a^b f(x)h(x)dx + \int_a^b g(x)h(x)dx = f \bullet h + g \bullet h,$$

(9.34) $$(\lambda f) \bullet g = \int_a^b \lambda f(x)g(x)dx = \lambda \int_a^b f(x)g(x)dx = \lambda(f \bullet g). \ \blacksquare$$

According to def.9.16 (norm for an inner product) and thm.9.2 (inner products define norms) we now define the norm associated with $f \bullet g = \int_a^b f(x)g(x)dx$*.*

Definition 9.18 (L_2–Norm for continuous functions)**.** Let $a, b \in \mathbb{R}$ such that $a < b$. Let $f \bullet g$ be the the following inner product on the space $\mathscr{C}([a,b], \mathbb{R})$ of all continuous functions $[a,b] \to \mathbb{R}$:

(9.35) $$f \bullet g := \int_a^b f(x)g(x)dx.$$

The associated norm

(9.36) $$\| \cdot \|_{L^2} : f \mapsto \|f\|_\bullet = \sqrt{\int_a^b f^2(x)dx}$$

is called the L^2**–norm**. of f. $\square$

We saw in def.9.15 that the Euclidean norm is the p–norm $\|\vec{x}\|_p = \Big(\sum_{j=1}^n {x_j}^p\Big)^{1/p}$ *for the special case* $p = 2$*. There is an analogue for the* L^2 *norm.*

Definition 9.19 (L^p–norms for $\mathbb{R}^n$)**.** Let $a, b \in \mathbb{R}$ such that $a < b$, $f \in \mathscr{C}([a,b], \mathbb{R})$, and $p \geqq 1$. Then

(9.37) $$f \mapsto \|f\|_{L^p} := \Big(\int_a^b |f(x)|^p\Big)^{1/p}$$

is a norm on $\mathscr{C}([a,b], \mathbb{R})$. This norm is called the L^p**–norm** of f. .

We saw that $\| \cdot \|_{L^p}$ is in fact a norm for $p = 2$. The proof for general $p \geqq 1$ is not given in this document. [80] $\square$

[80] You can find it, e.g., in [5] Haaser/Sullivan: Real Analysis.

9.2.3 The Inequalities of Young, Hoelder, and Minkowski $(\star)$

Proposition 9.14 (The p–norm in $\mathbb{R}^n$ is a norm)**.** *Let $p \in [1, \infty[$.*
Then the p-norm $\vec{x} \mapsto \|\vec{x}\|_p = \left(\sum_{j=1}^n |x_j|^p\right)^{1/p}$ is a norm in $\mathbb{R}^n$.

Proof:

***a**. Positive definiteness:*
Clearly, $\sum_{j=1}^n |x_j|^p \geqq 0$ because each term $|x_j|^p$ is non–negative, hence $\|\vec{x}\|_p = \sqrt{\sum_{j=1}^n |x_j|^p} \geqq 0$. Note that $\|\vec{x}\|_p = 0$ is only possible if $|x_j|^p = 0$ for all indices j, because, if $x_{j_0} \neq 0$ for some j_0 then $|x_{j_0}|^p > 0$, hence $\left(\|\vec{x}\|_p\right)^{1/p} \geqq |x_{j_0}|^p > 0$.

***b**. Absolute homogeneity:*
If $\lambda \in \mathbb{R}$ then

$$\|(\lambda\vec{x})\|_p = \Big(\sum_{j=1}^n (|\lambda|\,|x_j|)^p\Big)^{1/p} = \Big(|\lambda|^p \sum_{j=1}^n |x_j|^p\Big)^{1/p} = |\lambda|\Big(\sum_{j=1}^n |x_j|^p\Big)^{1/p} = |\lambda|\|\vec{x}\|_p$$

***c**. Triangle inequality for $p = 1$:*
It follows from $|x_j + y_j| \leqq |x_j| + |y_j|$ for all j that

$$\|\vec{x} + \vec{y}\|_1 = \sum_{j=1}^n |x_j + y_j| \leqq \sum_{j=1}^n |x_j| + \sum_{j=1}^n |y_j| = \|\vec{x}\|_1 + \|\vec{y}\|_1$$

***d**. Triangle inequality for $p > 1$:*
This is Minkowski's inequality for $(\mathbb{R}^n, \|\cdot\|_p)$ (thm.9.6 below).

That $\|\cdot\|_2$ satisfies the triangle inequality (i.e., $p = 2$) also follows independently from cor.9.1 on p.169. ■

Proposition 9.15 (The L^p–norm is a norm)**.** *Let $p \in [1, \infty[$ and let $a, b \in \mathbb{R}$ such that $a < b$. Then the L^p-norm $f \mapsto \|f\|_{L^p} = \left(\int_a^b |f(x)|^p\right)^{1/p}$ is a norm in $\mathscr{C}([a, b], \mathbb{R})$.*

Proof:

***a**. Positive definiteness:*
Follows from prop.9.12 on p.171 and the fact that $x \mapsto |f(x)|^p$ is a non–negative and continuous function.

***b**. Absolute homogeneity:*
If $\lambda \in \mathbb{R}$ then

$$\|(\lambda f)\|_{L^p} = \Big(\int_a^b (|\lambda|\,|f(x)|)^p\,dx\Big)^{1/p} = \Big(|\lambda|^p \int_a^b |f(x)|^p\,dx\Big)^{1/p} = |\lambda|\Big(\int_a^b |f(x)|^p\,dx\Big)^{1/p} = |\lambda|\|f\|_{L^p}.$$

c. Triangle inequality for $p = 1$:
It follows from $|f(x)+g(x)| \leqq |f(x)|+|g(x)|$ for all x that

$$\begin{aligned}\|f+g\|_{L^1} &= \int_a^b |f(x)+g(x)|\,dx \leqq \int_a^b (|f(x)| + |g(x)|)\,dx \\ &= \int_a^b |f(x)|\,dx + \int_a^b |g(x)|\,dx = \|\vec{x}\|_1 + \|\vec{y}\|_1.\end{aligned}$$

d. Triangle inequality for $p > 1$:
This is Minkowski's inequality for L^p–norms (thm.9.4 below).

That $\|\cdot\|_{L^2}$ satisfies the triangle inequality (i.e., $p = 2$) also follows independently from cor.9.1 on p.169. ■

Proposition 9.16 (Young's Inequality). *Let $a, b > 0$ and let $p, q > 1$ be **conjugate indices**, i.e.,*

(9.38) $$\frac{1}{p} + \frac{1}{q} = 1.$$

*Then the following holds **Young's inequality**:*

(9.39) $$ab \leqq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof:
***Step 1**: We show that $q - 1 = \frac{1}{p-1}$:*

(9.40) $$\begin{aligned}&\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow q+p = pq \Rightarrow q(1-p) = -p \Rightarrow q = \frac{p}{p-1} \\ &\Rightarrow q-1 = \frac{p-(p-1)}{p-1} = \frac{1}{p-1}.\end{aligned}$$

***Step 2**: The functions*

$$\varphi : [0,\infty[\to [0,\infty[;\quad x \mapsto x^{p-1} \quad \text{and} \quad \psi : [0,\infty[\to [0,\infty[;\quad y \mapsto y^{q-1}$$

are inverse to each other because we have

$$\psi\big(\varphi(x)\big) = \psi\big(x^{p-1}\big) = \big(x^{p-1}\big)^{q-1} \overset{(\star)}{=} \big(x^{p-1}\big)^{1/(p-1)} = x$$

($(\star)$ follows from step 1). We further have

$$\varphi\big(\psi(y)\big) = \varphi\big(y^{q-1}\big) = \big(y^{q-1}\big)^{p-1} \overset{(\star\star)}{=} \big(y^{q-1}\big)^{1/(q-1)} = y$$

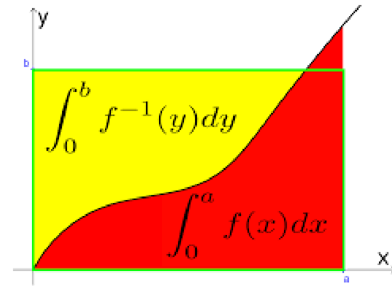
($(\star\star)$ again follows from step 1). Note that those two functions are continuous (actually, differentiable) and strictly increasing because $\varphi'(t) = (p-1)t^(p-2) > 0$ and $\psi'(t) = (q-1)t^(q-2) > 0$ for all $t \geqq 0$)). We further have $\varphi(0) = 0 = \psi(0)$.

***Step 3**: Let $f : [0,\infty[\to [0,\infty[$ be a continuous and strictly increasing (hence invertible) function such that $f(0) = 0$. Then the following is true for any two real numbers $a, b > 0$:*

(9.41) $$ab \leqq \int_0^a f(x)dx + \int_0^b f^{-1}(y)dy.$$

To prove this, we distinguish three cases. Either $b < f(a)$ or $b > f(a)$ or $b = f(a)$. The picture below shows what happens if $b < f(a)$: The rectangle ab is covered by the areas determined by the two integrals, but not all of the area of $\int_0^a f(x)dx$ is covered by the rectangle.

The picture to the right shows what happens if $b < f(a)$: The rectangle ab is covered by the areas determined by the two integrals, but not all of the area of $\int_0^a f(x)dx$ is covered by the rectangle.
Source: *https://brilliant.org/wiki/youngs-inequality/*



If $b > f(a)$ then the situation is similar, except that now not all of the area of $\int_0^b f^{-1}(y)dy$ is covered by the rectangle ab. Finally, if $b = f(a)$, the area covered by the two integrals matches the rectangle.

Step 4: *We now apply the above to the function $y = f(x) = x^{p-1}$. The inverse function is $x = f^{-1}(y) = y^{1/(p-1)} = y^{q-1}$ (see (9.40)). We integrate and obtain*

$$\int_0^a f(x)dx = \int_0^a x^{p-1} = \frac{x^p}{p}\Big|_0^a = \frac{a^p}{p}, \qquad \int_0^a f-1(y)dy = \int_0^a y^{q-1} = \frac{y^q}{q}\Big|_0^b = \frac{b^q}{q}.$$

Young's inequality (9.39) now follows from (9.41). ■

Theorem 9.3 (Hoelder's inequality for L^p–norms)**.** *Let $a, b \in \mathbb{R}$ such that $a < b$. Let $p, q > 1$ be conjugate indices, i.e.,*

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{9.42}$$

*Then **Hoelder's inequality** is true:*

$$\|fg\|_{L^1} \leqq \|f\|_{L^p}\|g\|_{L^q}, \quad i.e., \int_a^b |f(x)g(x)|dx \leqq \left(\int_a^b |f(x)|^p dx\right)^{1/p} \left(\int_a^b |g(x)|^q dx\right)^{1/q}. \tag{9.43}$$

Proof: We note that the composite function $x \mapsto |f(x)|^p$ is continuous, hence integrable, as the composite of the three continuous functions $x \mapsto f(x), y \mapsto |y|$, and $z \mapsto z^p$.

Note that $\|f\|_{L^p} = 0$ is only possible if $|f(x)|^p = 0$, i.e., $f(x) = 0$ for all x (see prop.9.12 on p.171). Likewise, $\|g\|_{L^q} = 0$ implies $g(x) = 0$ for all x. In either case, $\int_a^b f(x)g(x) = 0$ and (9.43) is trivially satisfied. So we may assume that both $\|f\|_{L^p} > 0$ and $\|g\|_{L^q} > 0$ For some fixed $x \in [a, b]$ let

$$A := \|f\|_{L^p}, \quad a_x := \frac{|f(x)|}{A}, \qquad B := \|g\|_{L^q}, \quad b_x := \frac{|g(x)|}{B}.$$

It follows from Young's inequality (9.39) that

$$a_x b_x \leqq \frac{a_x^p}{p} + \frac{b_x^q}{q}.$$

We integrate both sides of that inequality $\int_a^b \cdots dx$ *and obtain from the monotonicity of the integral (see example 4.19 on p.80) that*

$$\int_a^b a_x b_x dx \ \leqq \ \int_a^b \Big(\frac{a_x^{\,p}}{p} \ + \ \frac{b_x^{\,q}}{q}\Big)\, dx. \tag{9.44}$$

i.e.,

$$\begin{aligned} \frac{1}{AB}\int_a^b |f(x)g(x)|\, dx \ &\leqq \ \int_a^b \big(\frac{|f(x)|^p}{pA^p} \ + \ \frac{|g(x)|^q}{qB^q}\big)\, dx \\ &= \frac{1}{pA^p}\int_a^b |f(x)|^p\, dx \ + \ \frac{1}{qB^q}\int_a^b |g(x)|^q\, dx. \end{aligned} \tag{9.45}$$

We use

$$\int_a^b |f(x)|^p dx \ = \ (\|f\|_{L^p})^p, \quad \int_a^b |g(x)|^q dx \ = \ (\|g\|_{L^q})^q \tag{9.46}$$

in (9.45) and obtain

$$\frac{1}{AB}\int_a^b |f(x)g(x)| dx \ \leqq \ \frac{A^p}{pA^p} \ + \ \frac{B^q}{qB^q} \ = \ \frac{1}{p} \ + \ \frac{1}{q} \ = \ 1.$$

It follows from the definition of A *and* B *that*

$$\int_a^b |f(x)g(x)| dx \ \leqq \ AB \ = \ \|f\|_{L^p}\, \|g\|_{L^q}. \ \blacksquare$$

Theorem 9.4 (Minkowski's inequality for L^p–norms)**.** *Let* $a, b \in \mathbb{R}$ *such that* $a < b$ *and let* $p \in [1, \infty[$. *Then* ***Minkowski's inequality*** *is true:*

$$\|f+g\|_{L^p} \ \leqq \|f\|_{L^p} \ + \ \|g\|_{L^p}, \ \ i.e., \tag{9.47}$$

$$\Big(\int_a^b |f(x)+g(x)|^p\, dx\Big)^{1/p} \ \leqq \ \Big(\int_a^b |f(x)|^p\, dx\Big)^{1/p} \ + \ \Big(\int_a^b |g(x)|^p\, dx\Big)^{1/p}. \tag{9.48}$$

Proof: This follows for $p = 1$ *from part* ***c*** *of the proof of prop.9.15. We may assume that* $p > 1$. *Let* q *be the conjugate index to* p, *i.e.,*

$$\frac{1}{p} \ + \ \frac{1}{q} \ = \ 1, \ \textit{hence } (p-1)q \ = \ p \tag{9.49}$$

(see (9.40)). Let $a \leqq x \leqq b$. *Then*

$$|f(x)+g(x)|^p \ = |f(x)+g(x)|\, |f(x)+g(x)|^{p-1} \ \leqq \ |f(x)|\, |f(x)+g(x)|^{p-1} \ + \ |g(x)|\, |f(x)+g(x)|^{p-1}.$$

The last inequality follows from $|f(x)+g(x)| \leqq |f(x)| + |g(x)|$ *and* $|f(x)+g(x)|^{p-1} \geqq 0$. *We integrate and obtain*

$$\int_a^b |f(x)+g(x)|^p\, dx \ \leqq \ \int_a^b |f(x)|\, |f(x)+g(x)|^{p-1}\, dx \ + \ \int_a^b |g(x)|\, |f(x)+g(x)|^{p-1}\, dx. \tag{9.50}$$

We apply Hoelder's inequality to the first of the two integrals on the right hand side of (9.49) and obtain

$$\begin{aligned}\int_a^b (|f(x)|)\,\big(|f(x)+g(x)|^{p-1}\big)\,dx \;&\leqq \Big(\int_a^b (|f(x)|)^p\,dx\Big)^{1/p}\Big(\int_a^b \big(|f(x)+g(x)|^{p-1}\big)^q\,dx\Big)^{1/q}\\ &= \Big(\int_a^b |f(x)|^p\,dx\Big)^{1/p}\Big(\int_a^b |f(x)+g(x)|^{(p-1)q}\,dx\Big)^{1/q}\\ &= \Big(\int_a^b |f(x)|^p\,dx\Big)^{1/p}\Big(\int_a^b |f(x)+g(x)|^{p}\,dx\Big)^{1/q}.\end{aligned} \tag{9.51}$$

The last equality results from $(p-1)q = p$ (see (9.49)). Similarly, we obtain from the second integral on the right hand side of (9.49) the following:

$$\int_a^b (|g(x)|)\,\big(|f(x)+g(x)|^{p-1}\big)\,dx \;\leqq \Big(\int_a^b |g(x)|^p\,dx\Big)^{1/p}\Big(\int_a^b |f(x)+g(x)|^p\,dx\Big)^{1/q}. \tag{9.52}$$

We apply (9.51) and (9.52) to (9.50) and obtain

$$\begin{aligned}\int_a^b |f(x)+g(x)|^p\,dx \;&\leqq \Big(\int_a^b |f(x)|^p\,dx\Big)^{1/p}\Big(\int_a^b |f(x)+g(x)|^p\,dx\Big)^{1/q}\\ &\quad+\Big(\int_a^b |g(x)|^p\,dx\Big)^{1/p}\Big(\int_a^b |f(x)+g(x)|^p\,dx\Big)^{1/q}.\end{aligned} \tag{9.53}$$

Minkowski's inequality (9.48) is trivially satisfied if $\int_a^b |f(x)+g(x)|^p dx = 0$, so we may assume that $\int_a^b |f(x)+g(x)|^p dx > 0$. This allows us to divide each term in (9.53) by $\big(\int_a^b |f(x)+g(x)|^p dx\big)^{1/q}$. We obtain

$$\Big(\int_a^b |f(x)+g(x)|^p\,dx\Big)^{1-1/q} \;\leqq \Big(\int_a^b |f(x)|^p\,dx\Big)^{1/p} + \Big(\int_a^b |g(x)|^p\,dx\Big)^{1/p}. \tag{9.54}$$

Note that $1-\frac{1}{q} = \frac{1}{p}$ because $\frac{1}{q}+\frac{1}{p} = 1$, and (9.54) reads

$$\Big(\int_a^b |f(x)+g(x)|^p\,dx\Big)^{1/p} \;\leqq \Big(\int_a^b |f(x)|^p\,dx\Big)^{1/p} + \Big(\int_a^b |g(x)|^p\,dx\Big)^{1/p}. \;\blacksquare$$

Theorem 9.5 (Hoelder's inequality for the p–norms)**.** *Let $n \in \mathbb{N}$ and $\vec{x} = (x_1, \dots x_N), \vec{y} = (y_1, \dots y_N) \in \mathbb{R}^n$. Let $p, q > 1$ be conjugate indices, i.e.,*

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{9.55}$$

*Then **Hoelder's inequality** in $\mathbb{R}^n$ is true:*

$$\sum_{j=1}^n |x_j y_j| \;\leqq\; \|\vec{x}\|_p \|\vec{y}\|_q, \quad i.e., \sum_{j=1}^n |x_j y_j| \;\leqq\; \Big(\sum_{j=1}^n |x_j|^p\Big)^{1/p}\Big(\sum_{j=1}^n |y_j|^q\Big)^{1/q}. \tag{9.56}$$

Proof: Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. If $\vec{x} = 0$ or $\vec{y} = 0$ then $\sum_{j=1}^n |x_j y_j| = 0$ and (9.56) is trivially satisfied. We hence may assume that both $\vec{x} \neq 0$ and $\vec{y} \neq 0$.

*It follows from part **a** of the proof of prop.9.14 (positive definiteness of $\|\cdot\|_p$ for all p) on p.173 that $\|\vec{x}\|_p > 0$ and $\|\vec{y}\|_q > 0$ For some fixed index $1 \leqq j \leqq n$ let*

$$A := \|\vec{x}\|_p, \quad a_j := \frac{|x_j|}{A}, \qquad B := \|\vec{y}\|_q, \quad b_x := \frac{|y_j|}{B}.$$

It follows from Young's inequality (9.39) that

$$a_j b_j \;\leqq\; \frac{a_j^p}{p} \;+\; \frac{b_j^q}{q}.$$

We take sums $\sum_{j=1}^n \cdots$ of both sides of that inequality and obtain from the monotonicity of summation

$$\sum_{j=1}^n a_j b_j \;\leqq\; \sum_{j=1}^n \Big(\frac{(a_j)^p}{p} \;+\; \frac{(b_j)^q}{q}\Big), \tag{9.57}$$

i.e.,

$$\frac{1}{AB}\sum_{j=1}^n |x_j y_j| \;\leqq\; \sum_{j=1}^n \Big(\frac{|a_j|^p}{pA^p} \;+\; \frac{|b_j|^q}{qB^q}\Big) \;=\; \frac{1}{pA^p}\sum_{j=1}^n |a_j|^p \;+\; \frac{1}{qB^q}\sum_{j=1}^n |b_j|^q. \tag{9.58}$$

But

$$\sum_{j=1}^n |x_j|^p \;=\; (\|f\|_p)^p, \quad \sum_{j=1}^n |y_j|^q \;=\; (\|g\|_q)^q. \tag{9.59}$$

It follows from (9.58) that

$$\frac{1}{AB}\sum_{j=1}^n |x_j y_j| \;\leqq\; \frac{A^p}{pA^p} \;+\; \frac{B^q}{qB^q} \;=\; \frac{1}{p} \;+\; \frac{1}{q} \;=\; 1,$$

and we deduce from the definition of A and B that

$$\sum_{j=1}^n |x_j y_j| \;\leqq\; AB \;=\; \|\vec{x}\|_p\,\|\vec{x}\|_q. \ \blacksquare$$

Theorem 9.6 (Minkowski's inequality for $(\mathbb{R}^n, \|\cdot\|_p)$)**.** *Let $n \in \mathbb{N}$ and $\vec{x} = (x_1, \dots x_N)$, $\vec{y} = (y_1, \dots y_N) \in \mathbb{R}^n$. Let $p \in [1, \infty[$. Then **Minkowski's inequality for** $(\mathbb{R}^n, \|\cdot\|_p)$ is true:*

$$\|\vec{x} + \vec{y}\|_p \;\leqq \|\vec{x}\|_p \;+\; \|\vec{y}\|_p, \quad i.e., \tag{9.60}$$

$$\Big(\sum_j |x_j + y_j|^p\Big)^{1/p} \;\leqq\; \Big(\sum_j |f(x)|^p\Big)^{1/p} \;+\; \Big(\sum_j |g(x)|^p\Big)^{1/p}. \tag{9.61}$$

*Proof: This follows for $p = 1$ from part **c** of the proof of prop.9.15. We hence may assume that $p > 1$. Let q be the conjugate index to p, i.e.,*

$$\frac{1}{p} \;+\; \frac{1}{q} \;=\; 1, \ \textit{hence } (p-1)q \;=\; p \tag{9.62}$$

(see (9.40)). Let $a \leqq x \leqq b$. Then

$$|x_j + y_j|^p \; = |x_j + y_j|\,|x_j + y_j|^{p-1} \;\leqq\; |x_j|\,|x_j + y_j|^{p-1} \;+\; |y_j|\,|x_j + y_j|^{p-1}.$$

The last inequality follows from $|x_j + y_j| \leqq |x_j| + |y_j|$ and $|x_j + y_j|^{p-1} \geqq 0$. We sum and obtain

$$\sum_j |x_j + y_j|^p \;\leqq\; \sum_j |x_j|\,|x_j + y_j|^{p-1} \;+\; \sum_j |y_j|\,|x_j + y_j|^{p-1}. \tag{9.63}$$

Hoelder's inequality applied to the first of the two integrals on the right hand side of (9.63) yields

$$\begin{aligned} \sum_j \big(|x_j|\big)\,\big(|x_j + y_j|^{p-1}\big) \;&\leqq \Big(\sum_j \big(|x_j|\big)^p\Big)^{1/p} \Big(\sum_j \big(|x_j + y_j|^{p-1}\big)^q\Big)^{1/q} \\ &= \Big(\sum_j |x_j|^p\Big)^{1/p} \Big(\sum_j |x_j + y_j|^{(p-1)q}\Big)^{1/q} \\ &= \Big(\sum_j |x_j|^p\Big)^{1/p} \Big(\sum_j |x_j + y_j|^p\Big)^{1/q}. \end{aligned} \tag{9.64}$$

The last equality results from $(p-1)q = p$ (see (9.62)). Similarly, we obtain from the second integral on the right hand side of (9.63) the following:

$$\sum_j \big(|y_j|\big)\,\big(|x_j + y_j|^{p-1}\big) \;\leqq \Big(\sum_j |y_j|^p\Big)^{1/p} \Big(\sum_j |x_j + y_j|^p\Big)^{1/q}. \tag{9.65}$$

We apply (9.64) and (9.65) to (9.63) and obtain

$$\begin{aligned} \sum_j |x_j + y_j|^p \;&\leqq \Big(\sum_j |x_j|^p\Big)^{1/p} \Big(\sum_j |x_j + y_j|^p\Big)^{1/q} \\ &+ \Big(\sum_j |y_j|^p\Big)^{1/p} \Big(\sum_j |x_j + y_j|^p\Big)^{1/q}. \end{aligned} \tag{9.66}$$

Minkowski's inequality (9.61) is trivially satisfied if $\sum_j |x_j + y_j|^p = 0$, so we may assume that $\sum_j |x_j + y_j|^p > 0$. This allows us to divide each term in (9.66) by $\big(\sum_j |x_j + y_j|^p\big)^{1/q}$. We obtain

$$\Big(\sum_j |x_j + y_j|^p\Big)^{1-1/q} \;\leqq \Big(\sum_j |x_j|^p\Big)^{1/p} \;+\; \Big(\sum_j |y_j|^p\Big)^{1/p}. \tag{9.67}$$

Note that $1 - \frac{1}{q} = \frac{1}{p}$ because $\frac{1}{q} + \frac{1}{p} = 1$, and (9.67) reads

$$\Big(\sum_j |x_j + y_j|^p\Big)^{1/p} \;\leqq \Big(\sum_j |x_j|^p\Big)^{1/p} \;+\; \Big(\sum_j |y_j|^p\Big)^{1/p}. \;\blacksquare$$

9.3 Exercises for Ch.9

Exercise 9.1. Prove prop.9.10 on p.167: Let X be an arbitrary, non–empty set. Then the function $\|\cdot\|_\infty : \mathscr{B}(X,\mathbb{R}) \to \mathbb{R}_+, \quad h \to \|h\|_\infty = \sup\{|h(x)| : x \in X\}$ defines a norm. $\square$

Exercise 9.2. Prove prop.9.7 (Algebraic properties of the inner product) on p.165:

Let V be a vector space with inner product $\bullet(\cdot,\cdot)$. Let $a,b,x,y \in V$. Then

a. $(a+b)\bullet(x+y) = a\bullet x \;+\; b\bullet x \;+\; a\bullet y \;+\; b\bullet y$
b. $(x+y)\bullet(x+y) = x\bullet x \;+\; 2(x\bullet y) \;+\; y\bullet y$
c. $(x-y)\bullet(x-y) = x\bullet x \;-\; 2(x\bullet y) \;+\; y\bullet y$ $\square$

Exercise 9.3. Prove prop.9.11 on p.168: Let $(V,\|\cdot\|)$ be a normed vector space and let $\gamma > 0$. Let $p: \; V \to \mathbb{R}$ be defined as $p(x) := \gamma\|x\|$. Then p also is a norm. $\square$

Exercise 9.4. Prove that the p–norm (see def.9.15 on p.168) is a norm on $\mathbb{R}^n$ for the special case $p=1$:

$$\|\vec{x}\|_1 \;=\; \sum_{j=1}^{n} |x_j| \;\; \square$$

In chapter 10 on the topology of real numbers (p. 181) you will learn about metric spaces as a concept that generalizes the measurement of distance (or closeness, if you prefer) for the elements of a non–empty set.

10 Metric Spaces

There is a branch of Mathematics, called topology, which deals with the concept of closeness. The concept of limits of a sequence $(x_n)_n$ is based on closeness: The points of the sequence must get "arbitrarily close" to its limit as $n \to \infty$. Continuity of functions also can be phrased in terms of closeness: They map arbitrarily close elements of the domain to arbitrarily close elements of the codomain. In the most general setting Topology is about neighborhoods of a point without having the concept of measuring the distance of two points. We mostly won't deal with such a level of generality in this document. Instead we'll we'll focus on sets X that are equipped with a distance function.

10.1 The Topology of Metric Spaces (Study this!)

A metric is a real function of two arguments which associates with any two points $x, y \in X$ their "distance" $d(x, y)$.

It is clear how you measure the distance (or closeness, depending on your point of view) of two numbers x and y: you plot them on an x–axis where the distance between two consecutive integers is exactly one inch, grab a ruler and see what you get. Alternate approach: you compute the difference. For example, the distance between $x = 12.3$ and $y = 15$ is $x - y = 12.3 - 15 = -2.7$. Actually, we have a problem: There are situations where direction matters and a negative distance is one that goes into the opposite direction of a positive distance, but we do not want that in this context and understand the distance to be always non–negative, i.e.,

$$dist(x, y) = |y - x| = |x - y|$$

More importantly, you must forget what you learned in your in your science classes: "Never ever talk about a measure (such as distance or speed or volume) without clarifying its dimension". Is the speed measured in miles per hour our inches per second? Is the distance measured in inches or miles or micrometers? In the context of metric spaces we measure distance simply as a number, without any dimension attached to it. For the above example, you get

$$dist(12.3, 15) = |12.3 - 15| = 2.7.$$

In section 9.1.3 on p.152 it is shown in great detail that the distance between two two–dimensional vectors $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$ is

$$dist(\vec{v}, \vec{w}) = \sqrt{(w_1 - v_1)^2 + (w_2 - v_2)^2}$$

and the distance between two three–dimensional vectors $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ is

$$dist(\vec{v}, \vec{w}) = \sqrt{(w_1 - v_1)^2 + (w_2 - v_2)^2 + (w_3 - v_3)^2}.$$

In the next chapter we will generalize the concept of distance to more general objects.

10.1.1 Definition and Examples of Metric Spaces

Definition 10.1 (Metric spaces)**.** Let X be an arbitrary, non–empty set.
A **metric** on X is a real function of two arguments

$$d(\cdot, \cdot) : X \times X \to \mathbb{R}, \qquad (x, y) \mapsto d(x, y)$$

with the following three properties: [81]

(10.1a)	$d(x,y) \geqq 0 \quad \forall x,y \in X$ and $d(x,y) = 0 \ \Leftrightarrow\ x = y$	**positive definiteness**
(10.1b)	$d(x,y) = d(y,x) \quad \forall x,y \in X$	**symmetry**
(10.1c)	$d(x,z) \leqq d(x,y) + d(y,z) \quad \forall\, x,y,z \in X$	**triangle inequality**

The pair $(X, d(\cdot,\cdot))$, usually just written as (X,d), is called a **metric space**. We'll write X for short if it is clear which metric we are talking about. □

To appreciate that last sentence, you must understand that there can be more than one metric on X. See the examples below.

Remark 10.1 (Metric properties)**.** Let us quickly examine what those properties mean.

"Positive definite":	The distance is never negative and two items x and y have distance zero if and only if they are equal.
"symmetry":	the distance from x to y is no different to that from y to x. That may come as a surprise to you if you have learned in Physics about the distance from point a to point b being the vector $\vec{v}$ that starts in a and ends in b and which is the opposite of the vector $\vec{w}$ that starts in b and ends in a, i.e., $\vec{v} = -\vec{w}$. In this document we care only about size and not about direction.
"Triangle inequality":	If you directly walk from x to z then this will take less time than if you make a stopover at an intermediary y. □

Remark 10.2. Remark: Do not make the mistake and think of X as a set of numbers or vectors! For example, we might deal with

$$X \ := \ \{ \text{ all students who are currently taking this class } \}.$$

We can define the distance of any two students s_1 and s_2 as

$$d(s_1, s_2) = \begin{cases} 0 & \text{for } s_1 = s_2, \\ 1 & \text{for } s_1 \neq s_2. \end{cases}$$

We will learn later in this subchapter that the above function is called the discrete metric on X and satisfied indeed the definition of a metric. [82] □

The triangle inequality generalizes to more than two terms.

Proposition 10.1. *Let (X,d) be a metric space. Let $n \in \mathbb{N}$ and $x_1, x_2, \dots, x_n \in X$. Then*

$$d(x_1, x_n) \ \leqq \ \sum_{j=1}^{n-1} d(x_j, x_{j+1}) \ = \ d(x_1,x_2) \ + \ d(x_2,x_3) \ + \ d(x_{n-1},x_n). \tag{10.2}$$

[81] If you forgot the meaning of $X \times X$, it's time to review the definition of a cartesian product (def.4.1 in ch.4.1 on p.69.

[82] see def.10.3 on p.184 and prop.10.2 directly thereafter.

*The proof is left as exercise **??**.*

Before we give some examples of metric spaces, here is a theorem that tells you that a vector space with a norm (see def.9.14 on p.168), becomes a metric space as follows:

Theorem 10.1 (Norms define metric spaces)**.** *Let $(V, \|\cdot\|)$ be a normed vector space. Then the function*

$$d_{\|\cdot\|}(\cdot,\cdot) : V \times V \to \mathbb{R}_{\geqq 0}; \qquad (x,y) \mapsto d_{\|\cdot\|}(x,y) := \|y - x\| \tag{10.3}$$

defines a metric space $(V, d_{\|\cdot\|})$.

*The proof is left as exercise **??**.*

Definition 10.2 (Metric induced by a norm)**.** We say that the metric $d_{\|\cdot\|}(\cdot,\cdot)$ defined by (10.3) is **induced by the norm** $\|\cdot\|$. We also say that $d_{\|\cdot\|}(\cdot,\cdot)$ is **derived from the norm** $\|\cdot\|$ or that $d_{\|\cdot\|}(\cdot,\cdot)$ is **associated with the norm** $\|\cdot\|$. □

Here are some examples of metric spaces.

Example 10.1 (($\mathbb{R}$ with $d_{|\cdot|}(a,b) = |b-a|$)**.** According to thm.10.1 $(\mathbb{R}, d_{|\cdot|})$ is a metric space because the Euclidean norm $|\cdot|$ is a norm on $\mathbb{R} = \mathbb{R}^1$.

Here is a direct proof; It is obvious that if x, y are real numbers then the difference $x - y$, and hence its absolute value, is zero if and only if $x = y$ and that proves positive definiteness.

Symmetry follows from $d_{|\cdot|}(x,y) = |x-y| = |-(y-x)| = |y-x| = d_{|\cdot|}(y,x)$.

The triangle inequality for a metric follows from $|a+b| \leqq |a| + |b|$ (see prop.2.2 on p.17):

$$\begin{aligned} d_{|\cdot|}(x,z) &= |x-z| = |(x-y)-(z-y)| \\ &\leqq |x-y| + |z-y| = d_{|\cdot|}(x,y) + d_{|\cdot|}(z,y) = d_{|\cdot|}(x,y) + d_{|\cdot|}(y,z). \end{aligned} \quad \square$$

Example 10.2 (bounded real functions with $d_{\|\cdot\|_\infty} f, g) =$ sup–norm of $g(\cdot) - f(\cdot)$)**.**

$$d_{\|\cdot\|_\infty}(f,g) = \|g-f\|_\infty = \sup\{|g(x) - f(x)| : x \in X\} \tag{10.4}$$

is a metric on the set $\mathscr{B}(X, \mathbb{R})$ of all bounded real functions on X. This follows from thm.10.1 and prop.9.10 on p. 167, according to which $(\mathscr{B}(X,\mathbb{R}), \|\cdot\|_\infty)$ is a normed vector space. □

Example 10.3 (continuous real functions on [a,b])**.** with $d_{\|\cdot\|_{L^2}}(f,g) = L^2$–norm of $g(\cdot) - f(\cdot)$].

We will see in ch.10.1.2 on p.184 that $\|g - f\|_\infty$ is a good measure for the difference of the functions f and g and that an often even better measure is that of the area difference between their graphs which is given by the netric

$$d_{\|\cdot\|_{L^2}}(f,g) = \|g-f\|_{L^2} = \sqrt{\int_a^b \big(g(x) - f(x)\big)^2 dx}. \tag{10.5}$$

(see def.9.18 on p.172)

Example 10.4 ($\mathbb{R}^N$ with the Euclidean metric)**.**

$$d_{\|\cdot\|_2}(\vec{x}, \vec{y}) \;=\; \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \ldots + (y_N - x_N)^2} \;=\; \sqrt{\sum_{j=1}^{N} (y_j - x_j)^2}$$

This follows from the fact that the Euclidean norm is a norm on the vector space $\mathbb{R}^N$ (see cor.9.1 on p.169.) □

Just in case you think that all metrics are derived from norms, here is a counterexample.

Definition 10.3 (Discrete metric)**.** Let X be non–empty. Then the function

$$d(x, y) = \begin{cases} 0 & \text{for } x = y \\ 1 & \text{for } x \neq y \end{cases}$$

on $X \times X$ is called the **discrete metric** on X. □

The above definition makes sense because of the following proposition.

Proposition 10.2. *The discrete metric satisfies the properties of a metric.*

Proof: Obviously the function is non–negative and it is zero if and only if $x = y$. Symmetry is obvious too.

The triangle inequality $d(x, z) \leqq d(x, y) + d(y, z)$ is clear in the special case $x = z$. (Why?)

So let us assume $x \neq z$. But then $x \neq y$ or $y \neq z$ or both must be true. (Why?) That means that

$$d(x, z) = 1 \leqq d(x, y) + d(y, z)$$

and this proves the triangle inequality. ■

10.1.2 Measuring the Distance of Real Functions

How do we compare two functions? Let us make our lives easier: How do we compare two real functions $f(\cdot)$ and $g(\cdot)$? One answer is to look at a picture with the graphs of $f(\cdot)$ and $g(\cdot)$ and look at the shortest distance $|f(x) - g(x)|$ as you run through all x. That means that the distance between the functions $f(x) = x$ and $g(x) = x^2$ is zero because $f(1) = g(1) = 1$. The distance between $f(x) = x + 1$ and $g(x) = 0$ (the x–axis) is also zero because $f(-1) = g(-1) = 0$.

Do you really think this is a good way to measure closeness? You really do not want two items to have zero distance unless they coincide. It's a lot better to look for an argument x where the value $|f(x) - g(x)|$ is largest rather than smallest. Now we are ready for a proper definition.

Definition 10.4 (Maximal displacement distance between real functions)**.** Let X be an arbitrary, non-empty set and let $f(\cdot), g(\cdot) : X \to \mathbb{R}$ be two real functions on X. We define the **maximal**

displacement distance , also called the **sup–norm distance** or $\|\cdot\|_\infty$ **distance** , between $f(\cdot)$ and $g(\cdot)$ as

(10.6) $$d_\infty(f,g) = \|f(\cdot) - g(\cdot)\|_\infty = \sup\{|f(x) - g(x)| : x \in X\},$$

i.e., as the metric induced by the sup–norm on the set $\mathscr{B}(X,\mathbb{R})$ of all bounded real function on X. □

Remark 10.3. We have previously encountered the formula (10.6) in example 10.2 on p.183. We will see in prop.**??** on p.**??** of ch.10.3.1 on convergence of function sequences that the sup–norm induced metric is suitable to measure what will be called "uniform convergence" of real functions. As a metric, the distance measure of two functions f, g satisfies positive definiteness, symmetry and the triangle inequality. We have seen in other contexts what those properties mean.

"Positive definite": The distance is never negative and two functions $f(\cdot)$ and $g(\cdot)$ have distance zero if and only if they are equal, i.e., if and only if $f(x) = g(x)$ for each argument $x \in X$.

"Symmetry": the distance from $f(\cdot)$ to $g(\cdot)$ is no different than that from $g(\cdot)$ to $f(\cdot)$. Symmetry implies that you do **not** obtain a negative distance if you walk in the opposite direction.

"Triangle inequality": If you directly compare the maximum deviation between two functions $f(\cdot)$ and $h(\cdot)$ then this will never be more than than using an intermediary function $g(\cdot)$ and adding the distance between $f(\cdot)$ and $g(\cdot)$ to that between $g(\cdot)$ and $h(\cdot)$. □

Remark 10.4. The following picture illustrates the last definition. Plot the graphs of f and g as usual and find the the spot x_0 on the x–axis for which the difference $|f(x_0) - g(x_0)|$ (the length of the vertical line that connects the two points with coordinates $(x_0, f(x_0))$ and $(x_0, g(x_0))$) has the largest possible value. The domain of f and g is the subset of $\mathbb{R}$ that corresponds to the thick portion of the x–axis.

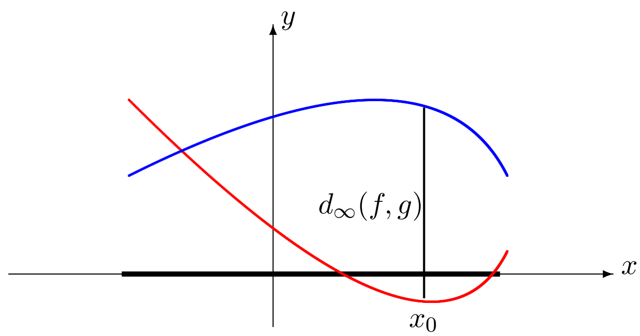


Figure 10.1: Distance of two real functions.

This figure allows you to visualize for a given $\delta > 0$ and $f : X \to \mathbb{R}$ the "δ–neighborhood" of $f(\cdot)$ defined as

(10.7) $$N_\delta(f) := \{g : X \to \mathbb{R} : d(_\infty f, g) < \delta\} = \{g(\cdot) : X \to \mathbb{R} : \sup_{x \in X}|f(x) - g(x)| < \delta\},$$

i.e., the set of all functions $g(\cdot)$ with distance less than δ from $f(\cdot)$.

You draw the graph of $f(\cdot)+\delta$ (the graph of $f(\cdot)$ shifted up north by the amount of δ) and the graph of $f(\cdot)-\delta$ (the graph of $f(\cdot)$ shifted down south by the amount of δ). Any function $g(\cdot)$ which stays completely inside this band, without actually touching it, belongs to the δ–neighborhood of $f(\cdot)$.

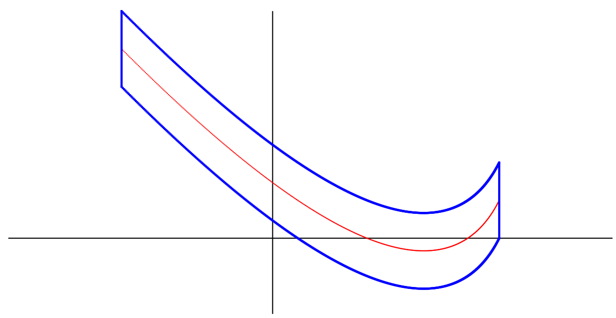


Figure 10.2: δ–neighborhood of a real function.

In other words, assuming that the domain A is a single, connected chunk and not a collection of several separate intervals, the δ–neighborhood of $f(\cdot)$ is a "band" whose contours are made up on the left and right by two vertical lines and on the top and bottom by two lines that look like the graph of $f(\cdot)$ itself but have been shifted up and down by the amount of δ. □

Definition 10.5 (Mean distances between real functions)**.** Let $a,b\in\mathbb{R}$ such that $a<b$ and let $f(\cdot),g(\cdot):X\to\mathbb{R}$ be two continuous real functions on X. We define the **mean square distance** between $f(\cdot)$ and $g(\cdot)$ on $[a,b]$ as

(10.8) $$d_{L^2}(f,g) \;:=d_{\|\cdot\|_{L^2}(f,g)} \;=\; \|g-f\|_{L^2} \;=\; \int_a^b (g(x)-f(x)^2dx,$$

i.e., as the metric induced by the L^2–norm on the set $\mathscr{C}_{\mathscr{B}}$([a,b], $\mathbb{R}$) of all continuous and bounded real function on $[a,b]$.

We further define the **mean distance** between $f(\cdot)$ and $g(\cdot)$ on $[a,b]$ as

(10.9) $$d_{L^1}(f,g) \;:=d_{\|\cdot\|_{L^1}(f,g)} \;=\; \|g-f\|_{L^1} \;=\; \int_a^b |g(x)-f(x|^2dx,$$

i.e., as the metric induced by the L^1–norm on the set $\mathscr{C}_{\mathscr{B}}$([a,b], $\mathbb{R}$). MAYBE preempt from ch. compactness: cont on [a,b] IS bounded

□

Remark 10.5. We saw in def.9.17, example 9.21 and def.9.18 on pp.170 that both

$$d_{L^1}(f,g) := d_{\|\cdot\|_{L^1}(f,g)} = \|g - f\|_{L^1} = \int_a^b |g(x) - f(x)|dx, \tag{10.10}$$

$$d_{L^2}(f,g) := d_{\|\cdot\|_{L^2}(f,g)} = \|g - f\|_{L^2} = \int_a^b (g(x) - f(x)^2 dx, \tag{10.11}$$

are often better suitable than the distance derived from the sup–norm to measure the distance of two functions. One of the drawbacks from a teaching perspective is that there is no picture like figure 10.2 to visualize the set of all functions with an L^1–distance or L^2–distance from a given function.

10.1.3 Neighborhoods and Open Sets

A. *Given a point $x_0 \in \mathbb{R}$ (a real number) and $\varepsilon > 0$, we can look at*

$$\begin{aligned} N_\varepsilon(x_0) = (x_0 - \varepsilon, x_0 + \varepsilon) &= \{x \in \mathbb{R} : x_0 - \varepsilon < x < x_0 + \varepsilon\} \\ &= \{x \in \mathbb{R} : d(x, x_0) = |x - x_0| < \varepsilon\} \end{aligned} \tag{10.12}$$

which is the set of all real numbers x with a distance to x_0 of strictly less than a number ε (the open interval with end points $x_0 - \varepsilon$ and $x_0 + \varepsilon$). (see example (10.1) on p.183).

B. *Given a point $\vec{x}_0 = (x_0, y_0) \in \mathbb{R}^2$ (a point in the xy–plane), we can look at*

$$\begin{aligned} N_\varepsilon(\vec{x}_0) &= \{\vec{x} \in \mathbb{R}^2 : \|\vec{x} - \vec{x}_0\| < \varepsilon\} \\ &= \{(x,y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 < \varepsilon^2\} \end{aligned} \tag{10.13}$$

which is the set of all points in the plane with a distance to $\vec{x}_0$ of strictly less than a number ε (the open disc around $\vec{x}_0$ with radius ε from which the points on the boundary (those with distance equal to ε) are excluded).

C. *Given a point $\vec{x}_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$ (a point in the 3–dimensional space), we can look at*

$$\begin{aligned} N_\varepsilon(\vec{x}_0) &= \{\vec{x} \in \mathbb{R}^3 : \|\vec{x} - \vec{x}_0\| < \varepsilon\} \\ &= \{(x,y,z) \in \mathbb{R}^3 : (\vec{x} - \vec{x}_0)^2 + (\vec{y} - \vec{y}_0)^2 + (\vec{z} - \vec{z}_0)^2 < \varepsilon^2\} \end{aligned} \tag{10.14}$$

which is the set of all points in space with a distance to $\vec{x}_0$ of strictly less than a number ε (the open ball around $\vec{x}_0$ with radius ε from which the points on the boundary (those with distance equal to ε) are excluded).

D. *Given a normed vector space $(V, \|\cdot\|)$ and a vector $x_0 \in V$, we can look at*

$$N_\varepsilon(x_0) = \{x \in V : \|x - x_0\| < \varepsilon\} \tag{10.15}$$

which is the set of all vectors in V with a distance to x_0 of strictly less than a number ε (the open set around x_0 with "radius" ε from which the points on the boundary (those with distance equal to ε) are excluded).

E. *Given a bounded real valued function $f \in \mathscr{B}(X, \mathbb{R})$, we can look at the sets $N_\varepsilon(f)$ $(\varepsilon > 0)$ defined in (10.7) on p.185, i.e., the set of all functions $g(\cdot)$ with distance less than ε from $f(\cdot)$.*

F. Given is a closed interval $[a,b]$ $(a,b \in \mathbb{R})$*. For a continuous (hence bounded) real valued function* $f \in \mathscr{B}([a,b],\mathbb{R})$*, we can look at the sets*

$$N_\varepsilon(f) \;=\; \{g \in \mathscr{B}([a,b],\mathbb{R}) : \|g - f\|_{L^2} < \varepsilon\}, \tag{10.16}$$

i.e., the set of all functions $g(\cdot)$ *such that* $\sqrt{\int_a^b \big(g(x) - f(x)\big)^2 dx} < \varepsilon$ *(see def.9.18 on p.172)*

There is one more item more general than neighborhoods of elements belonging to normed vector spaces, and that would be neighborhoods in metric spaces. We have arrived at the final definition:

Definition 10.6 (ε-Neighborhood)**.** Given a metric space (X,d) and an element $x_0 \in X$, let

$$N_\varepsilon(x_0) \;=\; \{x \in X : d(x,x_0) < \varepsilon\} \tag{10.17}$$

be the set of all elements of X with a distance to x_0 of strictly less than the number ε (the open set around x_0 with "radius" ε from which the points on the boundary (those with distance equal to ε) are excluded). We call $N_\varepsilon(x_0)$ the ε**–neighborhood** of x_0. □

The following should be intuitively clear: Look at any point $a \in N_\varepsilon(x_0)$*. You can find* $\delta > 0$ *such that the entire* δ*–neighborhood* $N_\delta(a)$ *of* a *is contained inside* $N_\varepsilon(x_0)$*. Just in case you do not trust your intuition, this is shown in prop. 10.4 just a little bit further down.*

It then follows that any $a \in N_\varepsilon(x_0)$ *is an interior point of* $N_\varepsilon(x_0)$ *in the following sense:*

Definition 10.7 (Interior point)**.** Given is a metric space (X,d).
An element $a \in A \subseteq X$ is called an **interior point** of A if we can find some $\varepsilon > 0$, however small it may be, so that $N_\varepsilon(a) \subseteq A$. □

Definition 10.8 (Open sets)**.** Given is a metric space (X,d).
A set all of whose members are interior points is called an **open set**. □

Proposition 10.3. *Let* (X,d) *be a metric space. Let* $x,y \in X$ *and* $\varepsilon > 0$ *such that* $y \in N_\varepsilon(x)$*.*

$$\text{If } \delta > 0 \text{ Then } N_\delta(y) \subseteq N_{\delta+\varepsilon}(x)$$

.

Proof: Let $z \in N_\delta(y)$*. Then*

$$d(z,x) \;\leqq\; d(z,y) + d(y,x) \;<\delta + \varepsilon.$$

In other words, each element z *of* $N_\delta(y)$ *is* $\delta + \varepsilon$*–close to* x*. Hence* $N_\delta(y) \subseteq N_{\delta+\varepsilon}(x)$*.* ■

Proposition 10.4. $N_\varepsilon(x_0)$ *is an open set*

It is worth while to examine the following proof [83] *closely because you can see how the triangle inequality is put to work.*

[83] A shorter proof can be given if the previous proposition is used.

$a \in N_\varepsilon(x_0)$ means that $\varepsilon - d(a, x_0) > 0$, say,

$$\varepsilon - d(a, x_0) = 2\delta \tag{10.18}$$

where $\delta > 0$. Let $b \in N_\delta(a)$. The claim is that any such b is an element of $N_\varepsilon(x_0)$. How so?

$$d(b, x_0) \leqq d(b, a) + d(a, x_0) < \delta + (\varepsilon - 2\delta) = \varepsilon - \delta < \varepsilon$$

In the above chain, the first inequality is a consequence of the triangle inequality. The second one reflects the fact that $b \in N_\delta(a)$ and uses (10.18).

We have proved that for any $b \in N_\delta(a)$ it is true that $b \in N_\varepsilon(x_0)$ hence $N_\delta(a) \subseteq N_\varepsilon(x_0)$.

We showed earlier on that any $a \in N_\varepsilon(x_0)$ is an interior point of $N_\varepsilon(x_0)$. ■

Definition 10.9 (Neighborhoods in Metric Spaces)**.** Let (X, d) be a metric space, $x_0 \in X$. Any open set that contains x_0 is called an **open neighborhood** of x_0. Any superset of an open neighborhood of x_0 is called a **neighborhood** of x_0. □

Remark 10.6 (Open neighborhoods are the important ones)**.** You will see that the important neighborhoods are the small ones, not the big ones. The definition above says that for any neighborhood A_x of a point $x \in X$ you can find an **open** neighborhood U_x of x such that $U_x \subseteq A_x$.

Because of this there are many propositions and theorems where you may assume that a neighborhood you deal with is open. □

Definition 10.10 (boundary points)**.** Let A be a subset of the metric space (X, d). $x \in X$ is called a **boundary point** of A if any neighborhood of x intersects both A and $A^{\complement}$. We write ∂A for the set of all boundary points of A and call this set the **boundary** of A. □

Theorem 10.2 (Metric spaces are topological spaces)**.** *The following is true about open sets of a metric space (X, d):*

(10.19a) *An arbitrary union $\bigcup_{i \in I} U_i$ of open sets U_i is open.*

(10.19b) *A finite intersection $U_1 \cap U_2 \cap \ldots \cap U_n$ $(n \in \mathbb{N})$ of open sets is open.*

(10.19c) *The entire set X is open and the empty set $\emptyset$ is open.*

Proof of a: Let $U := \bigcup_{i \in I} U_i$ and assume $x \in U$. We must show that x is an interior point of U. An element belongs to a union if and only if it belongs to at least one of the participating sets of the union. So there exists an index $i_0 \in I$ such that $x \in U_{i_0}$.

Because U_{i_0} is open, x is an interior point and we can find a suitable $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq U_{i_0}$. But $U_{i_0} \subseteq U$, hence $N_\varepsilon(x) \subseteq U$. It follows that x is interior point of U. But x was an arbitrary point of $U = \bigcup_{i \in I} U_i$ which therefore is shown to be an open set.

Proof of b: Let $x \in U := U_1 \cap U_2 \cap \ldots \cap U_n$. Then $x \in U_j$ for all $1 \leqq j \leqq n$ according to the definition of an intersection and it is inner point of all of them because they all are open sets. Hence, for each j there is a suitable $\varepsilon_j > 0$ such that $N_{\varepsilon_j}(x) \subseteq U_j$ Now define

$$\varepsilon := \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n\}$$

Then $\varepsilon > 0$ and [84]

$$N_\varepsilon(x) \subseteq N_{\varepsilon_j}(x) \subseteq U_j \ (1 \leqq j \leqq n), \quad \textit{hence} \quad N_\varepsilon(x) \subseteq \bigcap_{j=1}^{n} U_j.$$

We have shown that an arbitrary $x \in U$ is interior point of U and this proves part b.

Proof of c: First we deal with the set X. Choose any $x \in X$. No matter how small or big an $\varepsilon > 0$ you choose, $N_\varepsilon(x)$ is a subset of X. But then x is an inner point of X, so all members of x are inner points and this proves that X is open.
Now to the empty set $\emptyset$. You may have a hard time to accept the logic of this statement: All elements of $\emptyset$ are interior points. But remember, the premise "let $x \in X$" is always false and you may conclude from it whatever you please (see ch.3 (Logic). ■

This last theorem provides the underpinnings for the definition of abstract topological spaces which will be touched upon in ch.10.1.5 on p.192.

10.1.4 Convergence

You have already encountered the precise definition of the convergence of sequences of real numbers in ch.8.2. It is only a small step to generalize this concept to all metric spaces and therefore also to all normed vector spaces.

Definition 10.11 (convergence of sequences in metric spaces)**.** Given is a metric space (X, d).
We say that a sequence (x_n) of elements of X **converges** to $a \in X$ for $n \to \infty$ if almost all of the x_n will come arbitrarily close to a in the following sense:

Let δ be an arbitrarily small positive real number. Then there is a (possibly extremely large) integer n_0 such that all x_j belong to $N_\delta(a)$ just as long as $j \geqq n_0$. To say this another way: Given any number $\delta > 0$, however small, you can find an integer n_0 such that

$$d(a, x_j) < \delta \text{ for all } j \geqq n_0 \tag{10.20}$$

We write either of

$$a = \lim_{n\to\infty} x_n \qquad \text{or} \qquad x_n \to a \tag{10.21}$$

and we call a the **limit** of the sequence (x_n)

There is an equivalent way of expressing convergence towards a: No matter how small a neighborhood of a you choose: at most finitely many of the x_n will be located outside that neighborhood. □

Theorem 10.3 (Limits in metric spaces are uniquely determined)**.** *Let (X, d) be a metric space .*
Let $(x_n)_n$ be a convergent sequence in X Then its limit is uniquely determined.

[84] by the way, this is the exact spot where the proof breaks down if you deal with an infinite intersection of open sets: the minimum would have to be replaced by an infimum and there is no guarantee that it would be strictly larger than zero.

Proof: Otherwise there would be two different points $L_1, L_2 \in X$ such that both $\lim_{n\to\infty} x_n = L_1$ and $\lim_{n\to\infty} x_n = L_2$ Let $\varepsilon := d(L_1, L_2)/2$. There will be $N_1, N_2 \in \mathbb{N}$ such that

$$d(x_n, L_1) < \varepsilon \; \forall n \geqq N_1 \text{ and } d(x_n, L_2) < \varepsilon \; \forall n \geqq N_2.$$

It follows that, for $n \geqq \max(N_1, N_2)$, [85]

$$d(L_1, L_2) \;\leqq\; d(L_1, x_n) + d(x_n, L_2) \;<\; 2\varepsilon \;=\; d(L_1, L_2)$$

and we have reached a contradiction. ■

The following proposition shows that the limit behavior of a sequence is a property of its tail, i.e., it does not depend on the first finitely many indices.

Proposition 10.5. *Let x_n, y_n be two sequences in a metric space (X, d). Assume there is $K \in \mathbb{N}$ such that $x_n = y_n$ for all $n \geqq K$. Let $L \in X$ Then*

$$\lim_{n\to\infty} x_n = L \;\Leftrightarrow\; \lim_{n\to\infty} y_n = L.$$

Proof:
Left as an exercise. ■

Proposition 10.6. *Let x_n be a convergent sequence in a metric space (X, d) with limit $L \in E$. Let $K \in \mathbb{N}$. For $n \in \mathbb{N}$ let $y_n := x_{n+K}$. Then $\lim_{n\to\infty}(y_n)_n = L$.*

Proof:
Left as an exercise. ■

Remark 10.7.

The majority of mathematicians agrees that there is no "convergence to ∞" or "divergence to ∞". Rather, they will state that a sequence has the limit ∞. We follow this convention □

Remark 10.8 (Opposite of convergence)**.** Given a metric space (X, d), what is the opposite of $\lim_{k\to\infty} x_k = L$?
Beware! It is NOT the statement that $\lim_{k\to\infty} x_k \neq L$ because such a statement would mislead you to believe that such a limit exists, it just happens not to coincide with L

The correct answer: There exists some $\varepsilon > 0$ such that for **all** $N \in \mathbb{N}$ there exists some natural number $j = j(N)$ such that $j \geq N$ and $d(x_j, L) \geq \varepsilon$. □

It is easy to prove from the above remark the following:

Proposition 10.7 (Opposite of convergence)**.** *A sequence $(x_k)_k$ with values in (X, d) does not have $L \in X$ as its limit if and only if there exists some $\varepsilon > 0$ and $n_1 < n_2 < n_3 < \cdots \in \mathbb{N}$ such that $d(x_{n_j}, L) \geq \varepsilon$ for **all** j. In other words, we can find a subsequence $(x_{n_j})_j$ which completely stays out of some ε–neighborhood of L.*

The proof is left as exercise **??**.

[85] You could have used $N_1 + N_2$ instead. Do you see why?

10.1.5 Abstract Topological spaces

Theorem 10.2 on p.189 gives us a way of defining neighborhoods for sets which do not have a metric.

Definition 10.12 (Abstract topological spaces)**.** Let X be an arbitrary non-empty set and let $\mathfrak{U}$ be a set of subsets [86] of X whose members satisfy the properties a, b and c of (10.19) on p.189:

(10.22a) An arbitrary union $\bigcup_{i \in I} U_i$ of sets $U_i \in \mathfrak{U}$ belongs to $\mathfrak{U}$,

(10.22b) $U_1, U_2, \dots, U_n \in \mathfrak{U}\ (n \in \mathbb{N}) \quad \Rightarrow \quad U_1 \cap U_2 \cap \dots \cap U_n \in \mathfrak{U}$,

(10.22c) $X \in \mathfrak{U}$ and $\emptyset \in \mathfrak{U}$.

Then $(X, \mathfrak{U})$ is called a **topological space** The members of $\mathfrak{U}$ are called "open sets" of $(X, \mathfrak{U})$ and the collection $\mathfrak{U}$ of open sets is called the **topology** of X. □

Definition 10.13 (Topology induced by a metric)**.** Let (X, d) be a metric space and let $\mathfrak{U}_d$ be the set of open subsets of (X, d), i.e., all sets $U \subseteq X$ which consist of interior points only: for each $x \in U$ there exist $\varepsilon > 0$ such that

$$N_\varepsilon(x) \;=\; \{y \in X : d(x,y) < \varepsilon\} \;\subseteq\; U$$

(see (10.7) on p.188). We have seen in theorem (10.2) that those open sets satisfy the conditions of the previous definition. In other words, $(X, \mathfrak{U}_d)$ defines a topological space. We say that its topology is **induced by the metric** $d(\cdot,\cdot)$ or that it is **generated by the metric** $d(\cdot,\cdot)$. If there is no confusion about which metric we are talking about, we also simply speak about the **metric topology**.

Let X be a vector space with a norm $\|\cdot\|$. Remember that any norm defines a metric $d_{\|\cdot\|}(\cdot,\cdot)$ via $d_{\|\cdot\|}(x,y) = \|x - y\|$ (see (10.1) on p.183). Obviously, this norm defines open sets

$$\mathfrak{U}_{\|\cdot\|} \;:=\; \mathfrak{U}_{d_{\|\cdot\|}}$$

on X by means of this metric. We say that this topology is **induced by the norm** $\|\cdot\|$ or that it is **generated by the norm** $\|\cdot\|$. If there is no confusion about which norm we are talking about, we also simply speak about the **norm topology**. □

Example 10.5 (Discrete topology)**.** Let X be non–empty. Def.10.3 on p.184 gave the discrete metric as the function

$$d(x,y) \;=\; \begin{cases} 0 & \text{for } x = y \\ 1 & \text{for } x \neq y. \end{cases}$$

The associated topology is

$$\mathfrak{U}_d \;= 2^X \;=\; \{A : A \subseteq X\}.$$

Note that the discrete metric defines the biggest possible topology on X, i.e., the biggest possible collection of subsets of X whose members satisfy properties a, b, c of definition 10.12 on p.192. We call this topology the **discrete topology** of X. □

[86] We encountered subsets of 2^X with special properties previously when looking at rings of sets in def.2.13 (Rings and Algebras of Sets) on p.19.

Example 10.6 (Indiscrete topology)**.** Here is an example of a topology which is not generated by a metric. Let X be an arbitrary non–empty set and define $\mathfrak{U} := \{\emptyset, X\}$. Then $(X, \mathfrak{U})$ is a topological space. This is trivial because any intersection of members of $\mathfrak{U}$ is either $\emptyset$ (if at least one member is $\emptyset$) or X (if all members are X). Conversely, any union of members of $\mathfrak{U}$ is either $\emptyset$ (if all members are $\emptyset$) or X (if at least one member is X).

The topology $\{\emptyset, X\}$ is called the **indiscrete topology** of X. It is the smallest possible topology on X. □

Definition 10.14 (Base of the topology)**.** Let $(X, \mathfrak{U})$ be a topological space.

A subset $\mathfrak{B} \subseteq \mathfrak{U}$ of open sets is called a **base of the topology** if any nonempty open set U can be written as a union of elements of $\mathfrak{B}$:

(10.23)
$$U = \bigcup_{i \in I} B_i \quad (B_i \in \mathfrak{B} \text{ for all } i \in I)$$

where I is a suitable index set which of course will in general depend on U. □

We note that, because X itself is open, (10.23) implies that $X = \bigcup \left[B : B \in \mathfrak{B} \right]$.

Definition 10.15 (Neighborhoods and interior points in topological spaces)**.** Let $x \in X$ and $A \subseteq X$. It is not assumed that A be open. A is called a **neighborhood** of x and x is called an **inner point** or **interior point** of A if there exists an open set U such that

(10.24)
$$x \in U \subseteq A.$$

x is called an **exterior point** of A if there exists an open set U such that

(10.25)
$$x \in U \subseteq A^{\complement},$$

i.e., x is an inner point of $A^{\complement}$.

We call $ext(A) :=$ { all exterior points of A} the **open exterior** of A [87]

□

Definition 10.16 (Interior of a set in topological spaces)**.** Let $(X, \mathfrak{U})$ be a topological space and $A \subseteq X$. Let

(10.26)
$$A^o := \bigcup \left[U \in \mathfrak{U} : U \subseteq A \right]$$

be the union of all open subsets of A. We call A^o the **interior** of A. An alternate notation for A^o is $int(A)$.

It follows from def.10.12 (abstract topological spaces) on p.192 that A^o is an open set which is, as a union of subsets of A, also a subset of A. Because A^o is the union of all such sets, it follows that □

The interior A^o of A is the largest of all open subsets of A.

[87] Source: https://en.wikipedia.org/wiki/Interior_(topology)

Remark 10.9. Let $(X, \mathfrak{U})$ be a topological space and $A \subseteq X$. Note that the open exterior of A is

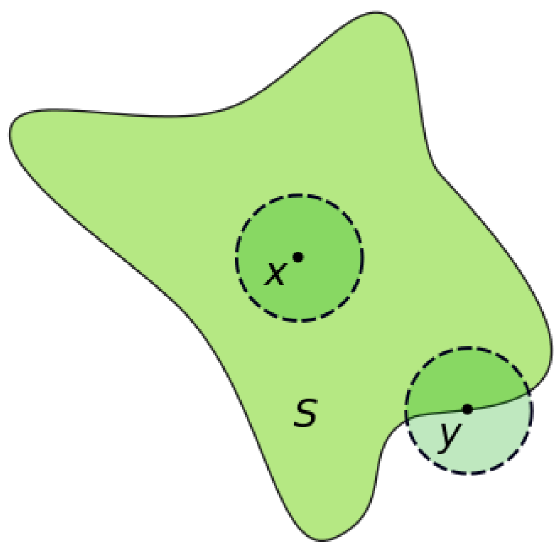
$$ext(A) \;=\; \overline{A^{\complement}}^{o}. \;\square \tag{10.27}$$

Remark 10.10. Let $(X, \mathfrak{U})$ be a topological space.

$$\text{If } A \subseteq B \subseteq X \text{ then } A^{o} \subseteq B^{o}. \;\square \tag{10.28}$$

Definition 10.17 (boundary points in topological spaces)**.** Let $(X, \mathfrak{U})$ be a topological space and $A \subseteq X$. Then $x \in X$ is called a **boundary point** of A if any neighborhood of x intersects both A and $A^{\complement}$. We write ∂A for the set of all boundary points of A and call this set the **boundary** of A.

Figure 10.3: Inner points v.s. boundary points.
Source: https://en.wikipedia.org/wiki/Interior_(topology)



We note that this definition is exactly the same as that given for metric spaces (compare def.10.10 on p.189). $\square$

Proposition 10.8. *Let (X, d) be a metric space and let $\mathfrak{B} := \{N_{1/k}(x) : x \in X, k \in \mathbb{N}\}$. Then $\mathfrak{B}$ is a base of the topology for the associated topological space $(X, \mathfrak{U}_d)$.*

The proof is left as exercise **??**.

Definition 10.18 (Second axiom of countability)**.** Let $(X, \mathfrak{U})$ be a topological space.
We say that X satisfies the **second axiom of countability** or X is **second countable** if we can find a countable base for $\mathfrak{U}$. $\square$

Theorem 10.4 (Euclidean space $\mathbb{R}^N$ is second countable)**.** *Let*

$$\mathfrak{B} \;:=\; \{\, N_{1/n}(q) : q \in \mathbb{Q}^N,\; n \in \mathbb{N} \,\}. \tag{10.29}$$

Here $\mathbb{Q}^N \;=\; \{q = (q_1, \dots, q_N) : q_j \in \mathbb{Q},\; 1 \leqq j \leqq N\}$ is the set of all points in $\mathbb{R}^N$ with rational coordinates. Then $\mathfrak{B}$ is a countable base.

Proof (outline): We recall from cor.7.5 on p. 120 that $\mathbb{Q}^N$ is countable. Let $U \in \mathfrak{U}$ be an arbitrary open set in X. Any $x \in U$ is inner point of U, hence we can find some $n_x \in \mathbb{N}$ such that the entire $\frac{3}{n_x}$–neighborhood $N_{3/n_x}(x)$ is contained within U. As any vector can be approximated by vectors with rational coordinates, there exists $q = q_x \in \mathbb{Q}^N$ such that $d(x, q_x) < 1/n_x$. Draw a picture and you see that both $x \in N_{1/n_x}(q_x)$ and $N_{1/n_x}(q_x) \subseteq N_{3/n_x}(x)$. In other words, we have

$$x \;\in\; N_{1/n_x}(q_x) \;\subseteq\; U$$

for all $x \in U$. But then

$$U \subseteq \bigcup \left[\, N_{1/n_x}(q_x) : x \in U \,\right] \;\subseteq U$$

and it follows that U is the (countable union of the sets $N_{1/n_x}(q_x)$. ■

We'll conclude this chapter with a summary of what we have learned about the classification of sets with a concept of closeness of points.

Remark 10.11 (Classification of topological spaces)**.** We have seen the following:

- **a.** $\mathbb{R}^N$ is an inner product spaces (see (9.8) on p.166).
- **b.** All inner product spaces are normed spaces (see (9.2) on p.169).
- **c.** All normed spaces are metric spaces (see (10.1) on p.183).
- **d.** All metric spaces are topological spaces. (see (10.12) on p.192 and (10.13) on p.192).

10.1.6 Neighborhood Bases ($\star$)

Note that this chapter is starred, hence optional.

Definition 10.19 (Neighborhood base)**.** Let $(X, \mathfrak{U})$ be a topological space.

The set of subsets of X

(10.30) $$\mathfrak{N}(x) \;:=\; \{A \subseteq X : A \text{ is a neighborhood of } x\}$$

is called the **neighborhood system of** x

Given a point $x \in X$, any subset $\mathfrak{B} := \mathfrak{B}(x) \subseteq \mathfrak{N}(x)$ of the neighborhood system of x is called a **neighborhood base of** x if it satisfies the following condition: For any $A \in \mathfrak{N}(x)$ you can find a $B \in \mathfrak{B}(x)$ such that $B \subseteq A$. □

In many propositions where proving closeness to some element is the issue, It often suffices to show that something is true for all sets that belong to a neighborhood base of x rather than having to show it for all neighborhoods of x. The reason is that often only the small neighborhoods matter and a neighborhood basis has "enough" of those.

Definition 10.20 (First axiom of countability)**.** Let $(X, \mathfrak{U})$ be a topological space.
We say that X satisfies the **first axiom of countability** or X is **first countable** if we can find for each $x \in X$ a countable neighborhood base. □

Proposition 10.9 (ε-neighborhoods are a base of the topology)**.** *Let (X, d) be a metric space. Then the set $\mathscr{B}_1 := \{N_\varepsilon(x) : x \in X, \varepsilon > 0\}$ is a base for the topology of (X, d) (see 10.14 on p.193) and the same is true for the "thinner" set $\mathscr{B}_2 := \{N_{1/n}(x) : x \in X, n \in \mathbb{N}\}$.*

Proof: To show that $\mathscr{B}_1$ (resp., $\mathscr{B}_2$) is a base we must prove that any open subset of X can be written as a union of (open) sets all of which belong to $\mathscr{B}_1$ (resp., $\mathscr{B}_2$). We prove this for $\mathscr{B}_2$.

Let $U \subseteq X$ be open. As any $x \in U$ is an interior point of U we can find some $\varepsilon = \varepsilon(x) > 0$ such that $N_{\varepsilon(x)}(x) \subseteq U$. We note that for any such $\varepsilon(x)$ there is $n(x) \in \mathbb{N}$ such that $1/n(x) \leqq \varepsilon(x)$.

We observe that $U \subseteq \bigcup \left[N_{1/n(x)}(x) : x \in U \right] \subseteq U$.

The first inclusion follows from the fact that $\{x\} \subseteq N_{1/n(x)}(x)$ for all $x \in U$ and the second inclusion follows from $N_{1/n(x)}(x) \subseteq U$ and the inclusion lemma (lemma 5.1 on p.97).

It follows that $U = \bigcup \left[N_{1/n(x)}(x) : x \in U \right]$ and we have managed to represent our open U as a union of elements of $\mathscr{B}_2$. This proves that $\mathscr{B}_2$ is a base for the topology of (X, d).

As $\mathscr{B}_2 \subseteq \mathscr{B}_1$ it follows that $\mathscr{B}_1$ also is such a base. ■

Theorem 10.5 (Metric spaces are first countable)**.** *Let (X, d) be a metric space. Then X is first countable.*

Proof (outline): For any $x \in X$ let

$$\mathfrak{B}(x) := \{ N_{1/n}(x) : n \in \mathbb{N} \}. \tag{10.31}$$

Then $\mathfrak{B}(x)$ is a neighborhood base of x. ■

10.1.7 Metric and Topological Subspaces

Definition 10.21 (Metric subspaces)**.** Given is a metric space (X, d) and a non–empty $A \subsetneq (X, d)$. Let $d\big|_{A \times A} : A \times A \to \mathbb{R}_{\geq 0}$ be the restriction $d\big|_{A \times A}(x, y) := d(x, y)(x, y \in A)$ of the metric d to $A \times A$ (see def.4.11 on p.86). It is trivial to verify that $(A, d\big|_{A \times A})$ is a metric space in the sense of def.10.1 on p.181. We call $(A, d\big|_{A \times A})$ a **metric subspace** of (X, d) and we call $d\big|_{A \times A}$ the **metric induced by** d or the **metric inherited from** (X, d). □

Remark 10.12.

Metric subspaces comes with their own collections of open and closed sets, neighborhoods, ε-neighborhoods, convergent sequences, ...
You must watch out when looking at statements and their proofs whether those concepts refer to the entire space (X, d) or to the subspace $(A, d\big|_{AxA})$. □

Notations 10.1.

a) Because the only difference between d and $d_{A \times A}$ is the domain, it is customary to write d instead of $d_{A \times A}$ to make formulas look simpler if doing so does not give rise to confusion.

b) We often shorten "open in $(A, d\big|_{A \times A})$" to "open in A", "closed in $(A, d\big|_{A \times A})$" to "closed in A", "convergent in $(A, d\big|_{A \times A})$" to "convergent in A", ... □

Definition 10.22 (Traces of sets in a metric subspace)**.** Let (X,d) be a metric space and $A \subseteq X$ a nonempty subset of X, viewed as a metric subspace $(A, d\big|_{A\times A})$ of (X,d) (see def.10.21 on p.196). Let $Q \subseteq X$. We call $Q \cap A$ the **trace** of Q in A.

For $\varepsilon > 0$ and $a \in A$ let $N_\varepsilon(a)$ be the ε-neighborhood of a (in (X,d)). We write

$$N_\varepsilon^A(a) \ := \ N_\varepsilon(a) \cap A,$$

i.e., $N_\varepsilon^A(a)$ is defined as the trace of $N_\varepsilon(a)$ in A. $\square$

Proposition 10.10 (Open sets in metric subspaces as traces of open sets in X)**.** *Let (X,d) be a metric space and $A \subseteq X$ a nonempty subset of X.*

a. *Let $\varepsilon > 0$ and $a \in A$. Then*

$$N_\varepsilon^A(a) \ = \ \{x \in A : d\big|_{A\times A}(x,a) < \varepsilon\}. \tag{10.32}$$

Because

$$N_\varepsilon^A(a) \ = \ N_\varepsilon(a) \cap A. \tag{10.33}$$

It follows that each ε-neighborhood in the subspace A is the trace of an ε-neighborhood in X.

b. *More generally, a set $U \subseteq A$ is open in A if and only if there is an open $V \subseteq$ in (X,d) such that*

$$U \ = \ V \cap A, \tag{10.34}$$

i.e., U is the trace of a set V which is open in X.

*Proof of **a**: First we prove (10.33). As $d\big|_{A\times A}$ is the restriction of d to $A \times A$ it follows that*

$$\begin{aligned} N_\varepsilon^A(a) \ &= N_\varepsilon(a) \cap A \ = \ \{x \in X : d(x,a) < \varepsilon\} \cap A \\ &= \{x \in A : d\big|_{A\times A} < \varepsilon\} \cap A \ = \ \{x \in A : d\big|_{A\times A} < \varepsilon\} \end{aligned}$$

*This finishes the proof of **a**.*

*Proof of **b**: First we show that if V is open in X then $U := V \cap A$ is open in the subspace A.*

Let $x \in U$. We must prove that x is an interior point of U with respect to $(A, d\big|_{A\times A})$ of (X,d).

Because $x \in V$ and V is open in X, there is $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq V$. It follows that

$N_\varepsilon^A(x) = N_\varepsilon(x) \cap A \subseteq V \cap A = U$ and $N_\varepsilon^A(x)$ is open in A, hence x is interior point of U with respect to the subspace $(A, d\big|_{A\times A})$.

Finally we prove that if $U \in A$ is open in A there is $V \subseteq X$ open in X such that $U = V \cap A$:

We can write $U \ = \ \bigcup \left[N_{\varepsilon(x)}^A(x) : x \in U\right]$ for suitable $\varepsilon(x) > 0$ (see the proof of prop.10.9 on p.195).

Let $V \ := \bigcup \left[N_{\varepsilon(x)}(x) : x \in U\right]$ we have

$$\begin{aligned} V \cap A \ &= A \cap \bigcup \left[N_{\varepsilon(x)}(x) : x \in U\right] \ = \ \bigcup \left[N_{\varepsilon(x)}(x) \cap A : x \in U\right] \\ &= \bigcup \left[N_{\varepsilon(x)}^A(x) : x \in U\right] \ = U \end{aligned}$$

(the second equalitity follows from prop.5.2 on p.98). This finishes the proof. ■

The last proposition justifies to define subspaces of abstract topological spaces as follows.

Definition 10.23 (Topological subspaces)**.** Let $(X, \mathfrak{U})$ be a topological space and $A \subseteq X$. We say that $V \subseteq A$ is **open in A** if V is the trace of an open set in X, i.e., if there is some $U \in \mathfrak{U}$ such that $V = U \cap A$. We denote the collection of all open sets in A as $\mathfrak{U}_A$, i.e.,

$$\mathfrak{U}_A = \{V \cap A : Y \in \mathfrak{U}\}.$$

We call $(A, \mathfrak{U}_A)$ a **topological subspace** or also just a **subspace** of $(X, \mathfrak{U})$.

We call $(\mathfrak{U}_A$ the **topology induced by** d or the **topology inherited from** (X, d). □

Proposition 10.11 (Topological subspaces are topological spaces)**.** *Let $(X, \mathfrak{U})$ be a topological space, $A \subseteq X$, and let $\mathfrak{U}_A$ be the collection of all open sets in A. Then $(A, \mathfrak{U}_A)$ is a topological space, i.e., it satisfies the definition def.10.12 on p.192 of an abstract topological space.*

Proof: Left as an exercise.

Remark 10.13 (Convergence does not extend to subspaces)**.** Let $A \subseteq (X, d)$ and $a_n \in A$.

a. Note that convergence of the sequence a_n in the space (X, d) (i.e., there exists $x \in X$ such that $x = \lim\limits_{n\to\infty} a_n$), does NOT imply convergence of the sequence in the space $(A, d\big|_{A\times A})$: Such is only the case if $x \in A$.

b. assume there exists $x \in X$ such that $\lim\limits_{n\to\infty} a_n = x$. We have the following <u>dichotomy</u>:
 Case 1 – $x \in A$: Then a_n converges in the subspace $(A, d\big|_{A\times A})$ (and in (X, d)).
 Case 2 – $x \in\in A^{\complement}$: Then a_n converges in (X, d) but not in $(A, d\big|_{A\times A})$.

□

10.1.8 Bounded Sets and Bounded Functions

Definition 10.24 (bounded sets)**.** Given is a subset A of a metric space (X, d). The **diameter** of A is defined as

(10.35) $$\operatorname{diam}(\emptyset) := 0, \qquad \operatorname{diam}(A) := \sup\{d(x, y) : x, y \in A\} \text{ if } A \neq \emptyset.$$

We call A a **bounded set** if $\operatorname{diam}(A) < \infty$. □

Proposition 10.12. *Given is a metric space (X, d) and a nonempty subset A. The following are equivalent:*

(10.36) ***a.*** *$\operatorname{diam}(A) < \infty$ i.e., A is bounded.*

(10.37) ***b.*** *There is a $\gamma > 0$ and $x_0 \in X$ such that $A \subseteq N_\gamma(x_0)$.*

(10.38) ***c.*** *For all $x \in X$ there is a $\gamma > 0$ such that $A \subseteq N_\gamma(x)$.*

*Proof of "**b** ⇒ **a**": For any $x, y \in A$ we have*

$$d(x, y) \leqq d(x, x_0) + d(x_0, y) \leqq 2\gamma$$

and it follows that diam$(A) \leqq 2\gamma$.

*Proof of "**a** $\Rightarrow$ **b**": Pick an arbitrary* $x_0 \in A$ *and let* $\gamma :=$ *diam*(A) *. Then*

$$y \in A \quad \to \quad d(x_0, y) \leqq \sup_{x \in A} d(x, y) \leqq \sup_{x,z \in A} d(x, z) = diam(A) = \gamma.$$

It follows that $A \subseteq N_\gamma(x_0)$.

*Proof of "**c** $\Rightarrow$ **a**": We pick an arbitrary* $x_0 \in A$ *which is possible as* A *is not empty. Then there is* $\gamma = \gamma(x_0)$ *such that* $A \subseteq N_\gamma(x_0)$*. For any* $y, z \in A$ *we then have*

$$d(y, z) \leqq d(y, x_0) + d(x_0, z) \leqq 2\gamma$$

and it follows that diam$(A) \leqq 2\gamma < \infty$.

*Proof of "**a** $\Rightarrow$ **c**": Given* $x \in X$*, pick an arbitrary* $x_0 \in A$ *and let* $\gamma := d(x, x_0) +$ *diam*(A)*. Then*

$$y \in A \quad \to \quad d(x, y) \leqq d(x, x_0) + d(x_0, y) \leqq d(x, x_0) + \sup_{u \in A} d(u, y)$$
$$\leqq d(x, x_0) + \sup_{u,z \in A} d(u, z) = d(x, x_0) + diam(A) = \gamma.$$

It follows that $A \subseteq N_\gamma(x)$. ■

10.1.9 Contact Points and Closed Sets

*If you look at any **closed interval*** $[a, b] = \{y \in \mathbb{R} : a \leqq y \leqq b\}$ *of real numbers, then all of its points are interior points, except for the end points a and b. On the other hand, a and b are contact points according to the following definition which makes sense for any abstract topological space.*

Definition 10.25 (Contact points)**.** Given is a topological space $(X, \mathfrak{U})$.
Let $A \subseteq X$ and $x \in X$ (x may or may not to belong to A). x is called a **contact point** [88] of A if

(10.39) $$A \cap N \neq \emptyset \text{ for any neighborhood } N \text{ of } x. \quad \square$$

The following theorem shows that we can characterize contact points of subsets of metric spaces by means of sequences.

Theorem 10.6 (Sequence criterion for contact points in metric spaces)**.** *Given is a metric space* (X, d)*. Let* $A \subseteq X$ *and* $x \in X$*. Then* x *is a contact point of* A *if and only if there exists a sequence* $x_1, x_2, x_3, \dots$ *of members of* A *which converges to* x*.*

Proof of "$\Rightarrow$": Let $x \in X$ *be such that* $N \cap A \neq \emptyset$ *for any neighborhood* N *of* x*. Let* $x_n \in N_{1/n}(x) \cap A$*. Such* x_n *exists because the neighborhood* $N_{1/n}(x)$ *has nonempty intersection with* A*.*

Given $\varepsilon > 0$*, let* $N \in \mathbb{N}$ *be chosen such that* $1/\varepsilon < N$*. This is possible because* $\mathbb{N}$ *is not bounded (above) in* $\mathbb{R}$*.*

[88] German: Berührungspunkt - see [9] Von Querenburg, p.21

For any $j \geqq N$ we obtain $d(x_j, x) < 1/j \leqq 1/N < \varepsilon$. This proves $x_n \to x$.

Proof of "$\Leftarrow$" Let $x \in X$ and assume there is $(x_n)_n$ such that $x_n \in A$ and $x_n \to x$.

We must show that if U_x is a (open) neighborhood of x then $U_x \cap A \neq \emptyset$. Let $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq U_x$.

It follows from $x_n \to x$ that there is $N = N(\varepsilon)$ such that $x_n \in N_\varepsilon(x)$ for all $n \geqq N$, especially, $x_N \in N_\varepsilon(x)$. By assumption, $x_N \in A$, hence $x_N \in N_\varepsilon(x) \cap A \subseteq U_x \cap A$, hence $U_x \cap A \neq \emptyset$. ■

Note 10.1. Note that any $a \in A$ is a contact point of A but not necessarily the other way around:

- **a.** Let $a \in A$. Then any neighborhood U_a of a contains a, hence $U_A \cap A$ is not empty, hence a is a contact point of A. This proves that any $a \in A$ is a contact point of A.
- **b.** Here is a counterexample which shows that the converse need not be true.
  Let $(X, d) := \mathbb{R}$ with the standard Euclidean metric and let A be the subset $]0, 1[$. We show now that 0 is a contact point of A.
  Any neighborhood A_0 of 0 contains for some small enough $\delta > 0$ the entire interval $]-\delta, \delta[$. Let $x := \min(\delta/2, 1/2)$.
  Clearly, $x \in]-\delta, \delta[\subseteq A_0$ and $x \in]0, 1[= A$.
  It follows that $x \in A \cap A_0$. As A_0 was an arbitrary neighborhood of 0, we have proved that 0 is a contact point of A, even though $0 \notin A$.
- **c.** The above counterexample can be proven much faster if the criterion for contact points in metric spaces is employed: Let $x_n := 1/n$ $(n \geqq 2)$ Then $x_n \in]0, 1[$ for all n and the sequence converges to 0. It follows that 0 is a contact point of $]0, 1[$. □

Note 10.2 (Contact points vs Limit points)**.** Besides contact points there also is the concept of a limit point. Here is the definition (see [7] Munkres, a standard book on topology):

Given is a metric space (X, d). Let $A \subseteq X$ and $a \in X$. a is called a **limit point** or **cluster point** or **point of accumulation** of A if any neighborhood U of a intersects A in at least one point <u>other than a</u>. This definition excludes "isolated points" [89] of A from being limit points of A. □

Definition 10.26 (Closed sets)**.** Given is a metric space (X, d) and a subset $A \subseteq X$. We call

$$\bar{A} := \{x \in X : x \text{ is a contact point of } A\} \tag{10.40}$$

the **closure** of A. A set that contains all its contact points is called a **closed set**. □

Remark 10.14. It follows from note 10.1.**a** that $A \subseteq \bar{A}$. □

Proposition 10.13. *The complement of an open set is closed.*

Proof of 10.13: Let A be an open set. Each point $a \in A$ is an interior point which can be surrounded by a δ-neighborhood $N_\delta(a)$ which, for small enough δ, will be entirely contained within A.

Let $B = A^{\complement} = X \setminus A$ and assume $x \in X$ is a contact point of B. We want to prove that B is a closed set, so we must show that $x \in B$.

We assume the opposite and show that this will lead to a contradiction. So let us assume that $x \notin B$.

[89] $a \in A$ is called an **isolated point** of A if there is a neighborhood U of a such that $U \cap A = \{a\}$.

That means, of course, that x belongs to B's complement which is A. But A is open, so x is an interior point of A. It follows that there is a neighborhood $N_\delta(x)$ surrounding x which is entirely contained in A, hence $N_\delta(x) \cap B = \emptyset$.

On the other hand we assumed that x is a contact point of B. This implies that $N_\delta(x)$ intersects B.

We have proved on one hand that $N_\delta(x) \cap B = \emptyset$ and on the other hand that there are points in B which also are contained in $N_\delta(x)$.

We have reached a contradiction. ■

Proposition 10.14. *The complement of a closed set is open.*

We will give two complete proofs of the above. The first one makes use of criterion for contact points (theorem 10.6) and works with sequences. The second proof is based on the definition of contact points and works with neighborhoods and interior points.

***a.** First proof of prop.10.14:*
Let A be closed set. Let $B = A^{\complement} = X \setminus A$. If B is not open then there must some be $b \in B$ which is not an interior point of B.

We show now that this assumption leads to a contradiction.

Because b is not an interior point of B, there is no δ-neighborhood, for whatever small δ, that entirely belongs to B. So, for each $j \in \mathbb{N}$, there is an $x_j \in N_{1/j}(b)$ which does not belong to B, i.e., $x_j \in A$.

We have constructed a sequence x_j which is entirely contained in A and which also converges to b. The latter is true because, for any j, all but finitely many members are contained in $N_{1/j}(b)$.

The closed set A contains all its contact points and it follows from the criterion for contact points that $b \in A$.

But we had assumed at the outset that $b \in B$ which is the complement of A and we have a contradiction. ■

***b.** Alternate proof of prop.10.14 which is entirely based on the concept of neighborhoods and interior points:*

Let A be closed set. Let $B = A^{\complement} = X \setminus A$. Let $b \in B$.

The closed set A contains all its contact points, so $b \notin A$ implies that b is not a contact point of A: according to def.10.25 there exists some neighborhood V of b such that $V \cap A = \emptyset$, i.e., $V \subseteq A^{\complement} = B$.

We have proved that an arbitrary $b \in B$ is an interior point of B, i.e., the complement B of the closed set A is open. ■

Theorem 10.7 (Open iff complement is closed)**.** *Let (X, d) be a metric space and $A \subseteq X$. Then A is open if and only if $A^{\complement}$ is closed.*

Proof: Immediate from prop.10.13 and prop.10.14 ■

Remark 10.15. a. We have seen that def.10.25 for contact points and hence def.10.26 for closed sets are entirely based on the concept of neighborhood which itself is entirely based on that of open sets. It follows that those two definitions make perfect sense not only in metric spaces but, more generally, in abstract topological spaces $(X, \mathfrak{U})$ which are characterized by the set $\mathfrak{U}$ of all open subsets of X (see def.10.12 on p.192).

b. Moreover the proof for prop.10.13 (complements of open sets are closed) and the first proof for prop.10.14 (complements of closed sets are open) are based on those definitions and do not employ specific properties of metric spaces either; theorem 10.7 also works for abstract topological spaces.

c. Matter of fact, many books define closed sets as the complements of open sets and only afterwards define contact points as we did. No surprise then that our definition of closed sets becomes their theorem: It is proved from those definitions that closed sets are exactly those that contain all their contact points. $\square$

Definition 10.27 (Contact points and closed Sets in topological spaces)**.** [90] Given is an abstract topological space $(X, \mathfrak{U})$.

Let $A \subseteq X$ and $x \in X$ (x may or may not to belong to A). x is called a **contact point** of A if

$$A \cap N \neq \emptyset \text{ for any neighborhood } N \text{ of } x.$$

We call

$$\bar{A} := \{x \in X : x \text{ is a contact point of } A\}$$

the **closure** of A. A set that contains all its contact points is called a **closed set**. $\square$

Remark 10.16. We note that $A \subseteq \bar{A}$: Let $a \in A$ and let V_a be a neighborhood of a. Because $a \in V_a$, we obtain $a \in V_a \cap A$, hence $V_a \cap A \neq \emptyset$, hence $a \in \bar{A}$.

It follows that A is closed if and only if $A = \bar{A}$ (which justifies the name "closure of A" for $\bar{A}$.) $\square$

Proposition 10.15. *Let $(X, \mathfrak{U})$ be a topological space and $A \subseteq B \subseteq X$. Then $\bar{A} \subseteq \bar{B}$.*

The proof is left as exercise **??**.

Proposition 10.16. *Let $(X, \mathfrak{U})$ be a topological space.*

The closed sets of X satisfy the following property:

(10.41)
- ***a.*** *An arbitrary intersection of closed sets is closed.*
- ***b.*** *A finite union of closed sets is closed.*
- ***c.*** *The entire set X is closed and $\emptyset$ is closed.*

*Proof of **a**: The proof is an easy consequence of De Morgan's law (the duality principle for sets) (see (5.1) on p.98). Observe that X is a universal set because all members U of $\mathfrak{U}$ and their complements $U^{\complement}$ are subsets of X.*

Let (C_α) be an arbitrary familiy of closed sets. Then $U_\alpha := C_\alpha^{\complement}$ is an open set for each α. Observe that $C_\alpha^{\complement} = U_\alpha$ because the complement of the complement of any set gives you back that set. Let $C := \bigcap\limits_\alpha C_\alpha$. Then

$$C^{\complement} = \Big(\bigcap_\alpha C_\alpha\Big)^{\complement} = \bigcup_\alpha C_\alpha^{\complement} = \bigcup_\alpha U_\alpha.$$

[90] see def.10.25 and def.10.26 for the following definitions in metric spaces.

*In other words $C^{\complement}$ is an arbitrary union of open sets which is open by the very definition of open sets of a topological space. We have proved **a**.*

*Proof of **b**: Let $C_1, C_2, \dots C_n$ be closed sets. Then $U_j := C_j^{\complement}$ is an open set for each j. Let $C := \bigcup\limits_{1 \leqq j \leqq n} C_j$. Then*

$$C^{\complement} = \Big(\bigcup_j C_j\Big)^{\complement} = \bigcap_j C_j^{\complement}. = \bigcap_j U_j$$

*Hence, $C^{\complement}$ is the intersection of finitely many open sets. This shows that $C^{\complement}$ is open, i.e., C is closed. We have proved **b**.*

*Proof of **c**: Trivial because*

$$X^{\complement} = \emptyset \qquad \text{and} \qquad \emptyset^{\complement} = X. \blacksquare$$

Proposition 10.17. *Let $(X, \mathfrak{U})$ be a topological space and $A \subseteq X$. Then*

$$\partial A = \bar{A} \cap \overline{A^{\complement}}, \tag{10.42}$$

i.e., $x \in X$ is a boundary point of A if and only if x is a contact point of both A and $A^{\complement}$.

Proof: Left as an exercise.

Proposition 10.18 (Minimality of the closure of a set)**.** *Let $(X, \mathfrak{U})$ be a topological space and $A \subseteq X$. Then*

$$\bar{A} = \bigcap \Big[C \supseteq A : C \text{ is closed} \Big]. \tag{10.43}$$

The closure $\bar{A}$ of A is the smallest of all closed supersets of A.

Proof: Let $\mathfrak{C} := \{C \supset A : C \text{ is closed }\}$ and let $F := \bigcap \mathfrak{C}$. We need to show that $\bar{A} = F$.

*It follows from prop.10.16.**a** that F is closed, hence $F = \bar{F}$. It follows from $C \supseteq A$ for all $C \in \mathfrak{C}$ that $F \supseteq A$, hence $F = \bar{F} \supseteq \bar{A}$.*

It remains to be shown that $F \subseteq \bar{A}$. It is true that $\bar{A} \in \mathfrak{C}$ because $\bar{A}$ is a closed set which contains A, hence $\bar{A} \supseteq \bigcap \mathfrak{C} = F$. (See prop.10.15 on p.202). $\blacksquare$

Proposition 10.19 (Closure of a set as a hull operator [91])**.** *Let $(X, \mathfrak{U})$ be a topological space. We can*

[91] This proposition states that the closure is a so–called **closure operator** which is defined to be a function

$$cl : 2^X \to 2^X; \qquad A \mapsto cl(A) := \bar{A}$$

on some abstract, non–empty set X (which need not be a topological space) such that the following are satisfied:

a. $(\emptyset) = \emptyset$, **b.** $A \subseteq cl(A)$, **c.** $cl(cl(A)) = cl(A)$, **d.** $cl(A \cup B) = cl(A) \cup cl(B)$.

It can be shown that if we define

$$\mathfrak{U} := \{A^{\complement} : cl(A) = A\}$$

then $(X, \mathfrak{U})$ satisfies the properties of a topological space.

think of the closure of sets as a function $^- : 2^X \to 2^X; \quad A \mapsto \bar{A}$*. This function has the following properties:*

a. $\bar{\emptyset} = \emptyset$*,* ***b.*** $A \subseteq \bar{A}$*,* ***c.*** $\bar{\bar{A}} = \bar{A}$*,* ***d.*** $\overline{A \cup B} = \bar{A} \cup \bar{B}$*.*

Proof: ***a*** *follows from* (10.41)*.**c*** *and* ***b*** *follows from remark 10.14.*

The proof of ***c*** *and* ***d*** *is left as exercise* **??***.*

10.1.10 Completeness in Metric Spaces

Often you are faced with a situation where you need to find a contact point a *and all you have is a sequence which behaves like one converging to a contact point in the sense of inequality 10.20 (page 190)*

Definition 10.28 (Cauchy sequences)**.** Given is a metric space (X,d).
A sequence (x_n) in X is called a **Cauchy sequence** [92] or, in short, it is Cauchy if it has the following property: Given any whatever small number $\varepsilon > 0$, you can find a (possibly very large) number n_0 such that

$$d(x_i, x_j) < \varepsilon \qquad \text{for all } i, j \geqq n_0 \tag{10.44}$$

This is called the **Cauchy criterion for convergence** of a sequence. □

Example 10.7 (Cauchy criterion for real numbers)**.** In $\mathbb{R}$ we have $d(x,y) = |x-y|$ and the Cauchy criterion requires for any given $\varepsilon > 0$ the existence of $n_0 \in \mathbb{N}$ such that

$$|x_i - x_j| < \varepsilon \qquad \text{for all } i, j \geqq n_0 \tag{10.45}$$ □

Proposition 10.20. *A Cauchy sequence in a metric space is bounded.*

Proof: Let $(x_n)_n$ *be a Cauchy sequence in a metric space* (X,d)*. There is* $N = N(1/2)$ *such that* $d(x_i,x_j) < 1/2$ *for all* $i,j \geqq N$*. In particular,* $d(x_i,x_N) < 1/2$*.*

Let $M := \max\{d(x_j,x_N) : j < N\}$*. We obtain for any two indices* $i,j \in \mathbb{N}$ *that*

$$d(x_i,x_j) \leqq d(x_i,x_N) + d(x_N,x_j).$$

$d(x_i,x_N)$ *is bounded by* M *in case that* $i < N$ *and by* $1/2$ *if* $i \geqq N$*; hence* $d(x_i,x_N) < 1/2 + M$*. We use the same reasoning to conclude that* $d(x_N,x_j) < 1/2 + M$ *and obtain* $d(x_i,x_j) < 1 + 2M$*. This proves the boundedness of* $(x_n)_n$*.* ■

The following theorem of the completeness of the set of all real numbers [93] *states that any Cauchy sequence converges to a real number. This is a big deal: To show that a sequence has a finite limit you need not provide the actual value of that limit. All you must show is that this sequence satisfies the Cauchy criterion. One can say that this preoccupation with proving existence rather than computing the actual value is one of the major points which distinguish Mathematics from applied Physics and the engineering disciplines.*

Here is the formal definition of a complete set in a metric space.

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[92] so named after the great french mathematician Augustin–Louis Cauchy (1789–1857) who contributed massively to the most fundamental ideas of Calculus.

[93] Remember the completeness axiom for $\mathbb{R}$ (axiom 8.1 on p.124) which states that any subset A of $\mathbb{R}$ which possesses upper bounds has a least upper bound (the supremum $\sup(A)$). This axiom was needed to establish the validity of thm.8.3 (Characterization of limits via limsup and liminf) on p.143, a theorem which will be used in this chapter to prove the completeness of $\mathbb{R}$ as a metric space.

Definition 10.29 (Completeness in metric spaces)**.** Given is a metric space (X, d). A subset $A \subseteq X$ is called **complete** if any Cauchy sequence (x_n) with elements in A converges to an element of A. □

Remark 10.17.

- **a.** In particular, X itself is complete if any Cauchy sequence in X converges.
- **b.** A is complete as a subset of (X, d) iff $((A, d\big|_{A \times A})$ is complete "in itself". □

Theorem 10.8 (Completeness of the real numbers)**.** *The following is true for the real numbers with the metric* $d(a, b) = |b - a|$ *but will in general be false for arbitrary metric spaces: Let* (x_n) *be a Cauchy sequence in* $\mathbb{R}$*. then there exists a real number* L *such that* $L = \lim_{n\to\infty} x_n$.

Proof: It follows from prop.10.20 that x_n *is bounded, hence* $(x_n)_n$ *possesses finite liminf and limsup.* [94] *We now show that* $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$.

Let $\varepsilon > 0$ *and* $N \in \mathbb{N}$ *such that* $|x_i - x_j| \leqq \varepsilon$ *for all* $i, j \geqq N$.

Let $T_n := \{x_j : j \geqq n\}$ *be the tail set of the sequence* $(x_n)_n$. *Let* $\alpha_N := \inf T_N, \beta_N := \sup T_N$.

There is some $i \geqq N$ *such that* $|x_i - \alpha_N| = x_i - \alpha_N \leqq \varepsilon$ *and there is some* $j \geqq N$ *such that* $|\beta_N - x_j| = \beta_N - x_j \leqq \varepsilon$. *It follows that*

$$0 \leqq \beta_N - \alpha_N = |\beta_N - \alpha_N| \leqq |\beta_N - x_j| + |x_j - x_i| + |x_i - \alpha_N| \leqq 3\varepsilon.$$

Further, if $k \geqq N$ *then* $T_k \subseteq T_N$*, hence* $\alpha_k \geqq \alpha_N$ *and* $\beta_k \leqq \beta_N$. *It follows that*

$$\beta_k - \alpha_k \leqq \beta_N - \alpha_N \leqq 3\varepsilon.$$

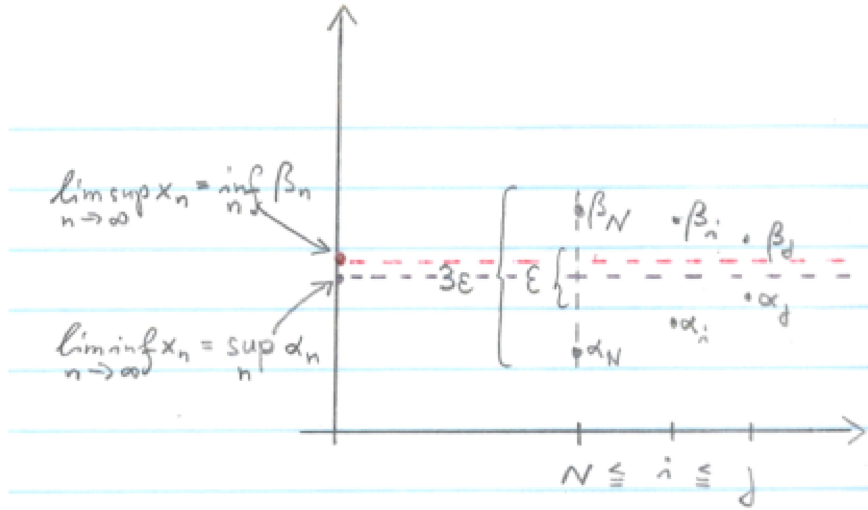
But then

$$0 \leqq \limsup_{k\to\infty} x_k - \liminf_{k\to\infty} x_k = \inf_k \beta_k - \sup_k \alpha_k \leqq \beta_N - \alpha_N \leqq 3\varepsilon.$$

$\varepsilon > 0$ *was arbitrary, hence* $\limsup_{k\to\infty} x_k = \liminf_{k\to\infty} x_k$.

[94] See ch.8.1 (Minima, Maxima, Infima and Suprema).

Figure 10.4: ε-δ continuity



Part 3: It follows from theorem 8.3 on p.143 that the sequence $(x_n)_n$ converges to $L := \limsup_{k\to\infty} x_k$ and the proof is finished. ■

Now that the completeness of $\mathbb{R}$ has been established, it is not very difficult to see that N–dimensional space $\mathbb{R}^N$ also is complete.

Theorem 10.9 (Completeness of $\mathbb{R}^N$)**.** *Let $(\vec{x}_n)$ be a Cauchy sequence in $\mathbb{R}^N$.*
Then there exists a vector $\vec{a} \in \mathbb{R}^N$ such that $\vec{a} = \lim_{n\to\infty} \vec{x}_n$.

Proof (outline): Let $\vec{x}_j = (x_{j,1}, x_{j,2}, \dots, x_{j,N})$ be Cauchy in $\mathbb{R}^N$. For fixed k, each coordinate sequence $(x_{j,k})_j$ is Cauchy because, if $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that if $i, j \geqq K$ then $\|\vec{x}_i - \vec{x}_j\|_2 < \varepsilon$. Hence

$$|x_{i,k} - x_{j,k}| \;=\; \sqrt{|x_{i,k} - x_{j,k}|^2} \;\leqq\; \sqrt{\sum_{k=1}^{N} |x_{i,k} - x_{j,k}|^2} \;=\; \|\vec{x}_i - \vec{x}_j\|_2 \;<\; \varepsilon.$$

It follows from the completeness of $\mathbb{R}$ as a metric space that there exist real numbers

$$a_1, a_2, a_3, \dots, a_N \qquad \text{such that } \; a_k \;=\; \lim_{n\to\infty} x_{n,k} \; (1 \leqq k \leqq N).$$

For a given number ε we can find natural numbers $n_{0,1}, n_{0,2}, \dots, n_{0,N}$ such that

$$|x_{n,k} - a_k| \;<\; \frac{\varepsilon}{N} \quad \text{for all } n \geqq n_{0,j} \text{ and for all } 1 \leqq k \leqq N.$$

Let $n^\star := \max(n_{0,1}, n_{0,2}, \dots, n_{0,N})$. It follows that

$$d(\vec{x}_n - \vec{a}) \;=\; \sqrt{\sum_{k=1}^{N} |x_{n,k} - a_k|^2} \;\leqq\; \sqrt{N \cdot \left(\frac{\varepsilon}{N}\right)^2} \;=\; \frac{\varepsilon}{\sqrt{N}} \;\leqq\; \varepsilon \qquad \text{for all } n \geqq n^\star. \;\blacksquare$$

Example 10.8 (Approximation of decimals)**.** The following illustrates Cauchy sequences and completeness in $\mathbb{R}$. Take any real number $x \geqq 0$ and write it as a decimal:

$$x \;=\; m \;+\; \sum_{j=1}^{\infty} d_j \cdot 10^{-j} \quad (m \in \mathbb{Z},\; d_j \in \{0,1,2,\dots,9\})$$

As was explained in (2.9) on (p.16), anything that can be written as a decimal number is a real number. Let's say, x starts out on the left as

$$x \;=\; 258.1408926584207531\dots$$

We define as x_k the leftmost part of x, truncated k digits after the decimal points:

$$x_1 \;=\; 258.1, \quad x_2 \;=\; 258.14, \quad x_3 \;=\; 258.140, \quad x_4 \;=\; 258.1408, \quad x_5 \;=\; 258.14089, \quad \dots$$

We further define y_k the leftmost part of x, truncated k digits after the decimal points, but the rightmost digit incremented by 1 (where you then might obtain a carry-over to the left when you add 1 to 9)

$$y_1 \;=\; 258.2, \quad y_2 \;=\; 258.15, \quad y_3 \;=\; 258.141, \quad y_4 \;=\; 258.1409, \quad y_5 \;=\; 258.14090, \quad \dots$$

then the sequence (x_n) is non-decreasing: $x_{n+1} \geqq x_n$ for all n and the sequence (y_n) is non-increasing: $y_{n+1} \leqq y_n$ for all n. We have the sandwiching property: $x_n \leqq x \leqq y_n$ for all n. Both sequences are Cauchy because

$$d(x_{n+i}, x_{n+j}) \;= |x_{n+i} - x_{n+j}| \;\leqq\; 10^{-n} \;\to\; 0 \quad (n \to\; \infty),$$

$$d(y_{n+i}, y_{n+j}) \;= |y_{n+i} - y_{n+j}| \;\leqq\; 10^{-n} \;\to\; 0 \quad (n \to\; \infty).$$

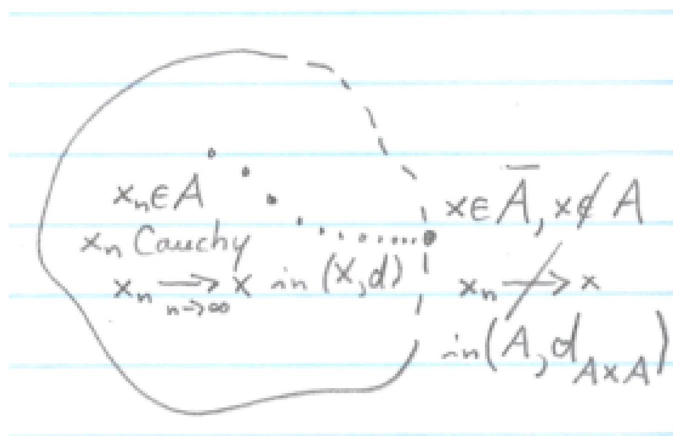
It follows that both sequences have limits. It is obvious that $x \;=\; \lim\limits_{n\to\infty} x_n \;=\; \lim\limits_{m\to\infty} y_m$.

What just has been illustrated is that there a natural way to construct for a given $x \in \mathbb{R}$ Cauchy sequences that converge towards x. The completeness principle states that the reverse is true: For any Cauchy sequence there is an element x towards which the sequence converges. $\square$

We won't really talk about completeness in general until the chapter on compact spaces. Just to mention one of the simplest facts about completeness:

Theorem 10.10 (Complete sets are closed)**.** *Any complete subset of a metric space is closed.*

Figure 10.5: complete $\Rightarrow$ closed



Proof: *Let (X, d) be a metric space and $A \subseteq X$. Let $a \in X$ be a contact point of A. The theorem is proved if we can show that $a \in A$.*

a) *We employ prop.10.25 on p.199: A point $x \in X$ is a contact point of A if and only if $A \cap V \neq \emptyset$ for any neighborhood V of x.*

Let $m \in \mathbb{N}$. Then $N_{1/m}(a)$ is a neighborhood of the contact point a, hence hence $A \bigcap N_{1/m}(a) \neq \emptyset$ and we can pick a point from this intersection which we name x_m.

b) *We prove next that $(x_m)_m$ is Cauchy. Let $\varepsilon > 0$ and let $N \in \mathbb{N}$ be such that $N > 1/\varepsilon$. if $j \in \mathbb{N}$ and $k \in \mathbb{N}$ both exceed N then*

$$d(x_j, x_k) \leqq d(x_j, a) + d(a, x_k) \leqq \frac{1}{j} + \frac{1}{k} \leqq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

It follows that the sequence (x_j) is Cauchy.

c) *Because A is complete, this sequence must converge to some $b \in A$. But b cannot be different from a Otherwise we could "separate" a and b by two disjoint neighborhoods: choose the open ρ–balls $N_\rho(a)$ and $N_\rho(b)$ where ρ is one half the distance between a and b (see the proof of thm.10.3 on p.190).*

Only finitely many of the x_n are allowed to be outside $N_\rho(a)$ and the same is true for $N_\rho(b)$. That is a contradiction and it follows that $b = a$, i.e., $a \in A$.

d) *We summarize: if a is a contact point of A then $a \in A$. It follows that A is closed.* ■

The following is the reverse of thm.10.10.

Theorem 10.11 (Closed subsets of a complete space are complete)**.** *Let (X, d) be a complete metric space and let $A \subseteq X$ be closed. Then A is complete, i.e., the metric subspace $(A, d\big|_{A \times A})$ is complete.*

Proof: *Let $(x_n)_n$ be a Cauchy sequence in A. We must show that there is $a \in A$ such that $x_n \to a$. (x_n) also is Cauchy in X because the Cauchy criterion is entirely specified in terms of members of the sequence (x_n).*

Because X is complete there exists $x \in X$ such that $x_n \to x$. All x_n belong to A. According to thm.10.6 (Sequence criterion for contact points in metric spaces), x is a contact point of A.

As the set A is assumed to be closed, it contains all its contact points. It follows that $x \in A$, i.e., the arbitrary Cauchy sequence (x_n) in A converges to an element of A. We conclude that A is complete. ■

Theorem 10.12 (Convergent sequences are Cauchy). *Let $(x_n)_n$ be a convergent sequence in a metric space (X, d). Then $(x_n)_n$ is Cauchy.*

Proof: Let $L \in X$ and $x_n \to L$. Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that

$$k \geqq N \quad \Rightarrow \quad d(x_k, L) \; < \; \varepsilon/2. \tag{10.46}$$

It follows from (10.46) that, for any $i, j \geqq N$,

$$i, j \geqq N \quad \Rightarrow \quad d(x_i, x_j) \; \leqq \; d(x_i, L) + d(L, x_j) \; < \; \varepsilon/2 + \varepsilon/2 \; = \; \varepsilon.$$

It follows that the sequence satisfies (10.44) of the definition of a Cauchy sequence (def. 10.28 on p.204). ■

Proposition 10.21. *Let $(x_n)_n$ be a Cauchy sequence in a metric space (X, d) such that some subsequence x_{n_j} converges to a limit L. Then ANY subsequence of $(x_n)_n$ converges to L. This is true in particular for the full sequence $x_1, x_2, \dots$, i.e., $(x_n)_n$ is a convergent sequence.*

Proof: Let $n_1 < n_2 < n_3 \dots$ be such that x_{n_j} converges to L. For $k \in \mathbb{N}$ let $y_k := x_{n_k}$.

Let $\varepsilon > 0$. Convergence $y_j \to L$ implies that there is $N \in \mathbb{N}$ such that

$$d(y_j, L) < \varepsilon/2 \text{ for all } j \geqq N. \tag{10.47}$$

Because (x_j) is Cauchy there also exists $N' \in \mathbb{N}$ such that

$$d(x_i, x_j) < \varepsilon/2 \text{ for all } i, j \geqq N'. \tag{10.48}$$

Let $K := \max(n_N, N')$ and $j \geqq K$. Then

$$d(x_j, L) \; \leqq \; d(x_j, y_K) + d(y_K, L)$$

It follows from $n_K \geqq K$ and $j \geqq K$ and (10.48) that $d(x_j, y_K) = d(x_j, x_{n_K}) < \varepsilon/2$ and it follows from (10.47) that $d(y_K, L) < \varepsilon/2$. We conclude that $d(x_j, L) < \varepsilon$ for all $j \geqq K$ and this proves convergence $x_j \to L$. ■

10.2 Continuity (Study this!)

10.2.1 Definition and Characterizations of Continuous Functions

We have briefly discussed in ch.8.2 on p.129. the continuity of functions with arguments and values in $\mathbb{R}$*. We now extend this definition to functions that map from metric spaces to metric spaces.*

Definition 10.30 (Sequence continuity)**.** Given are two metric spaces (X, d_1) and (Y, d_2). Let $A \subseteq X$, $x_0 \in A$ and let $f : A \to Y$ be a mapping from A to Y. We say that f is **sequence continuous at** $\boldsymbol{x_0}$ and we write

$$\lim_{x \to x_0} f(x) = f(x_0) \tag{10.49}$$

if the following is true for any sequence (x_n) with values in A:

$$\text{if } x_n \to x_0 \text{ then } f(x_n) \to f(x_0). \tag{10.50}$$

In other words, the following must be true for any sequence (x_n) in A and $x_0 \in A$:

$$\lim_{n\to\infty} x_n = x_0 \quad \Rightarrow \quad \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x_0). \tag{10.51}$$

We say that f is **sequence continuous** if f is sequence continuous at x_0 for all $x_0 \in A$. □

Remark 10.18. Important points to notice:

a) It is not enough for the above to be true for some sequences that converge to x_0. Rather, it must be true for all such sequences!

b) We restrict our universe to the domain A of f: x_0 and the entire sequence $(x_n)_{n\in\mathbb{N}}$ must belong to A because there must be function values for all x-values. In other words, f is continuous at $x_0 \in A$ if and only if f is continuous at x_0 in the metric subspace $(A, d\big|_{A\times A})$. □

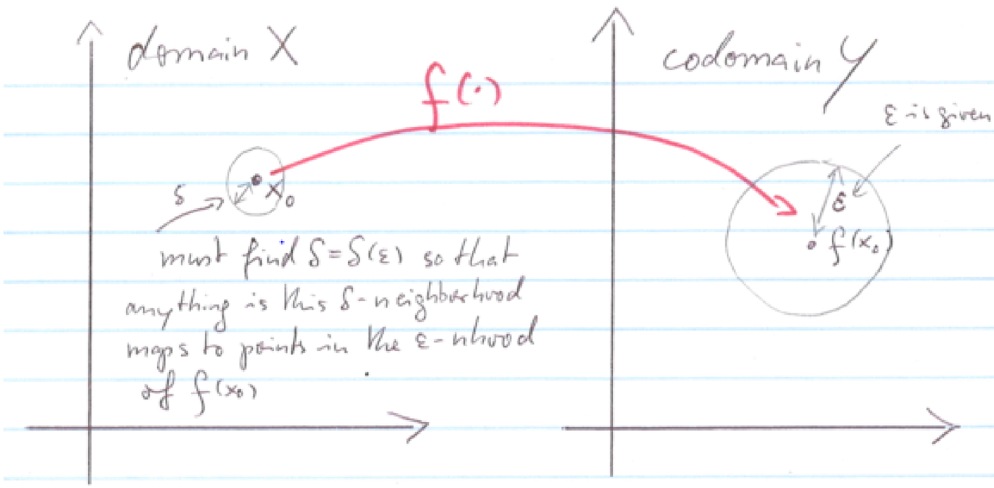
Definition 10.31 (ε-δ continuity)**.** Given are two metric spaces (X, d_1) and (Y, d_2). Let $A \subseteq X$, $x_0 \in A$ and let $f(\cdot) : A \to Y$ be a mapping from A to Y. We say that $f(\cdot)$ is ε**-**δ **continuous at** $\boldsymbol{x_0}$ if the following is true: For any (whatever small) $\varepsilon > 0$ there exists $\delta > 0$ such that either one of the following equivalent statements is satisfied:

$$f\big(N_\delta(x_0) \cap A\big) \subseteq N_\varepsilon(f(x_0)), \tag{10.52}$$

$$d_1(x, x_0) < \delta \Rightarrow d_2(f(x), f(x_0)) < \varepsilon \text{ for all } x \in A. \tag{10.53}$$

We say that $f(\cdot)$ is ε**-**δ **continuous** if $f(\cdot)$ is ε-δ continuous at a for all $a \in A$. □

Figure 10.6: ε-δ continuity



Remark 10.19. We recall from thm.10.32 on p.197 that

$$N_\delta \cap A \;=\; N_\delta^A(a) \;=\; \{x \in A : d\big|_{A\times A}(x,a) < \delta\}. \tag{10.54}$$

Hence, (10.52) states that f is ε-δ continuous at x_0 if and only if $f\big(N_\delta^A(x_0)\big) \subseteq N_\varepsilon(f(x_0))$. $\square$

Theorem 10.13 (ε-δ characterization of continuity)**.** *Let (X,d_1) and (Y,d_2) be two metric spaces. Let $A \subseteq X$, $x_0 \in A$ and let $f(\cdot) : A \to Y$ be a mapping from A to Y. Then f is sequence continuous at x_0 if and only if f is ε-δ continuous at x_0.*

In particular f is sequence continuous if and only if f is ε-δ continuous.

a) *⇒: Proof that sequence continuity implies ε-δ-continuity:*
We assume to the contrary that there exists some function f which is sequence continuous but not ε-δ-continuous at x_0, i.e., there exists some $\varepsilon > 0$ such that neither (10.52) nor the equivalent (10.53) is true for any $\delta > 0$.

a.1. *In other words, No matter how small a δ we choose, there is at least one $x = x(\delta) \in A$ such that $d_1(x,x_0) < \delta$ but $d_2(f(x),f(x_0)) \geqq \varepsilon$. In particular we obtain for $\delta := 1/m (m \in \mathbb{N})$ that*

$$\textit{there exists some } x_m \in N_{1/m}(x_0) \cap A; \textit{such that}; d_2(f(x_m),f(x_0)) \geqq \varepsilon. \tag{10.55}$$

a.2. *We now show that the sequence $(x_m)_{m\in\mathbb{N}}$ converges to x_0: Let $\gamma > 0$. There exists $N = N(\gamma) \in \mathbb{N}$ so big that $N > 1/\gamma$, i.e., $1/N < \gamma$. As $x_m \in N_{1/m}(x_0)$, we obtain for all $m \geqq N$ that*

$$d_1(x_m,x_0) < 1/m \leqq 1/N < \gamma.$$

This proves that $x_m \to x_0$.

a.3. *Clearly, the sequence $\big(f(x_m)\big)_{m\in\mathbb{N}}$ does not converge to $f(x_0)$ as that requires $d_2(f(x_m),f(x_0)) < \varepsilon$ for all sufficiently big m, contrary to (10.55) which implies that there is not even one such m. In other words, the function f is not sequence continuous, contrary to our assumption. We have our contradiction.*

***b*) ⇐**: *Proof that ε-δ-continuity implies sequence continuity:*
Let $x_n \to x_0$. Let $y_n := f(x_n)$ and $y := f(x_0)$. We must prove that $y_n \to y$ as $n \to \infty$.

b.1. *Let $\varepsilon > 0$. We can find $\delta > 0$ such that (10.52) and hence (10.53) is satisfied. We assumed that $x_n \to x_0$. Hence there exists $N := N(\delta) \in \mathbb{N}$ such that $d_1(x_n, x_0) < \delta$ for all $n \geqq N$.*

b.2. *It follows from (10.53) that $d_2(y_n, y) = d_2(f(x_n), f(x_0)) < \varepsilon$ for all $n \geqq N$. In other words, $y_n \to y$ as $n \to \infty$ and the proof of "⇐" is finished.*

Finally, the equivalence f is ε-δ continuous ⇔ f is ε-δ continuous. ■

*From now on we can use the terms "ε-δ continuous at x_0" and "sequence continuous at x_0" interchangeably for functions between metric spaces and we will simply speak about **continuity of f at** x_0.*

We saw in the ε-δ continuity definition of a function with metric spaces for both domain and codomain and the subsequent remark 10.19 that continuity of $f : (A, d_1\big|_{A \times A}) \to (Y, d_2)$ in $x_0 \in A$ was equivalent to demanding that for any ε–neighborhood of $f(x_0)$ there is a δ–neighborhood of x_0 such that

$$f\big(N_\delta^A(x_0)\big) \subseteq N_\varepsilon(f(x_0)).$$

Considering that any neighborhood of a point z in a metric space contains a γ–neighborhood of z for suitably small γ, the following theorem should not come as a surprise.

Theorem 10.14 (Neighborhood characterization of continuity)**.** *Let (X, d_1) and (Y, d_2) be two metric spaces. Let $A \subseteq X$, $x_0 \in A$, and let $f(\cdot) : A \to Y$ be a mapping from A to Y. Then f is continuous at x_0 if and only if for any neighborhood $V_{f(x_0)}$ of $f(x_0)$ there exists a neighborhood U_{x_0} of x_0 in the metric space (X, d_1) such that*

$$f(U_{x_0} \cap A) \subseteq V_{f(x_0)}. \tag{10.56}$$

Equivalently, this can be stated in terms of the subspace $(A, d_1\big|_{A \times A})$ as follows. for any neighborhood $V_{f(x_0)}$ of $f(x_0)$ there exists a neighborhood $U_{x_0}^A$ of x_0 in the metric space $(A, d_1\big|_{A \times A})$ such that

$$f(U_{x_0}^A) \subseteq V_{f(x_0)}. \tag{10.57}$$

***a*) ⇒)**: *Assume that f is continuous, i.e., ε-δ continuous at a. Let $V_{f(x_0)}$ be a neighborhood of x_0.*

Then $f(x_0)$ is interior point of $V_{f(x_0)}$ and we can find suitable $\varepsilon > 0$ such that $N_\varepsilon(f(x_0)) \subseteq V_{f(x_0)}$. ε-δ continuity at a implies the existence of $\delta > 0$ such that $f\big(N_\delta(x_0) \cap A\big) \subseteq V_{f(x_0)}$.

This proves both (10.56) (choose $U_{x_0} := N_\delta(x_0)$) and (10.57) (choose $U_{x_0}^A := N_\delta(x_0) \cap A$).

***b*) ⇐)**: *Assume that (10.56) is satisfied for any arbitrary neighborhood $V_{f(x_0)}$ of $f(x_0)$.*

Let $\varepsilon > 0$. We need to show that there exists $\delta > 0$ such that

$$f(N_\delta(x_0) \cap A) \subseteq N_\varepsilon(f(x_0)). \tag{10.58}$$

$N_\varepsilon(f(x_0))$ is a neighborhood of $f(x_0)$. It follows from (10.56) that there exists a neighborhood U_{x_0} of x_0 such that

$$f(U_{x_0} \cap A) \subseteq N_\varepsilon(f(x_0)). \tag{10.59}$$

x_0 is interior point of any of its neighborhoods. In particular, it is interior to U_{x_0}.

Accordingly, there exists $\delta > 0$ such that $N_\delta(x_0) \subseteq U_{x_0}$, hence $N_\delta(x_0) \cap A \subseteq U_{x_0} \cap A$. It follows from the monotonicity of the direct image $\Gamma \mapsto f(\Gamma)$ that

$$f\big(N_\delta(x_0) \cap A\big) \subseteq f\big(U_{x_0} \cap A\big) \subseteq N_\varepsilon(f(x_0)). \tag{10.60}$$

The second inclusion relation follows from (10.59). We have proved the existence of $\delta > 0$ such that (10.58) is satisfied. This finishes the proof of "$\Leftarrow$)". ■

The last theorem allows us to define continuity for functions between abstract topological spaces.

Definition 10.32 (Continuity for topological spaces)**.** Given are two topological spaces $(X, \mathfrak{U}_1)$ and $(Y, \mathfrak{U}_2)$. Let $A \subseteq X$, $x_0 \in A$ and let $f(\cdot) : A \to Y$ be a mapping from A to Y. We say that $f(\cdot)$ is **continuous at x_0** if the following is true:

For any neighborhood $V_{f(x_0)}$ of $f(x_0)$ there exists a neighborhood U_{x_0} of x_0 in the topological space $(X, \mathfrak{U}_1)$ such that

$$f(U_{x_0} \cap A) \subseteq V_{f(x_0)}. \tag{10.61}$$

Equivalently, this can be stated in terms of the subspace $(A, {\mathfrak{U}_1}_A)$ as follows. For any neighborhood $V_{f(x_0)}$ of $f(x_0)$ there exists a neighborhood $U^A_{x_0}$ of x_0 in $(A, {\mathfrak{U}_1}_A)$ such that

$$f(U^A_{x_0}) \subseteq V_{f(x_0)}. \tag{10.62}$$

We say that $f(\cdot)$ is **continuous** if $f(\cdot)$ is continuous at a for all $a \in A$. □

[1] B/G: Art of Proof defines in appendix A, p.136, continuity of a function f as follows: "f^{-1}(open) $=$ open". The following proposition proves that their definition coincides with the one given here: the validity of (10.52) for all $x_0 \in X$.

***a)** In the interest of simplicity f now is defined on all of X and not just on some subset A of X. Note that the general case of $f : A \to Y$ is covered by replacing (X, d_1) with $(A, d_1\big|_{A \times A})$, i.e., we deal with $f : (A, d_1\big|_{A \times A}) \to (Y, d_2)$.*

***b)** Also note that this next proposition addresses continuity of f for **all** $x \in X$ and **not** at a specific x_0.*

Proposition 10.22 ("f^{-1}(open) $=$ open" continuity)**.** *Let $(X, \mathfrak{U})$ and $(Y, \mathfrak{V})$ be two topological spaces and let $f(\cdot) : X \to Y$ be a mapping from X to Y. Then $f(\cdot)$ is continuous if and only if the following is true: Let V be an open subset of Y. Then the inverse image $f^{-1}(V)$ is open in X.* [95]

Proof of "$\Rightarrow$": Let V be an open set in Y. Let $U := f^{-1}(V)$, $a \in U$ and $b := f(a)$. Then $b \in V$ by the definition of inverse images. b is inner point of the open set V and there is according to def.10.32 a neighborhood U_a of $a \in X$ such that $f(U_a) \subseteq V$.

We conclude from the monotonicity of direct and inverse images and prop.6.1 on p.107 that

$$U_a \subseteq f^{-1}\big(f(U_a)\big) \subseteq f^{-1}(V) = U.$$

[95] This is easily extended to $f : A \to Y$ ($\emptyset \neq A \subseteq X$) by demanding that $f^{-1}(V)$ is open in $(A, \mathfrak{U}_A)$.

It follows that the arbitrarily chosen $a \in U$ is an interior point of U and this proves that U is open.

Proof of "⇐": We now assume that all inverse images of open sets in Y are open in X.

Let $a \in X, b = f(a)$, and let V_b be a neighborhood of b. Any neighborhood of b contains an open neighborhood of b, hence we may assume that V_b is open. We are done if we can find an open neighborhood U_a of a such that

$$f(U_a) \subseteq V_b \tag{10.63}$$

Let $U_a := f^{-1}(V_b)$. Then U_a is open as the inverse image of the open set V_b It follows from the monotonicity of direct and inverse images and prop.6.6 on p.108 that

$$f(U) = f\big(f^{-1}(V_b)\big) = V_b \cap f(X) \subseteq V_b.$$

We have proved (10.63) ■

Remark 10.20 (continuity for real functions of real numbers)**.** Let $(X, d_1) = (Y, d_2) = \mathbb{R}$. In this case equation (10.53) on p.210 becomes

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

See thm.8.1 on p.136. □

Proposition 10.23 (continuity of the identity mapping)**.** *Let $X, d)$ be a metric space and*

$$id_X : X \to X; \qquad x \mapsto x$$

be the identity function on X. Then id_X is continuous.

Proof: Given any $\varepsilon > 0$, let $\delta := \varepsilon$. Let $x, y \in X$. Assume that $d(x, y) < \delta$. Then

$$d(id_X(x), id_X(y)) = d(x, y) < \delta = \varepsilon$$

and we have satisfied condition (10.53) *of the $\varepsilon - \delta$ characterization of continuity.* [96] *This proves that the identity mapping is continuous.* ■

10.2.2 Continuity of Constants and Sums and Products

For all the following, unless stated differently, let (X, d) be a metric space and $A \subseteq X$, $A \neq \emptyset$. Let

$$f : A \to \mathbb{R}, \qquad g : A \to \mathbb{R}$$

be two real functions which both are continuous in a point $x_0 \in A$. Moreover, let $a, b \in \mathbb{R}$. You can think of any fixed number a as a function

$$a(\cdot) : A \to \mathbb{R}; x \mapsto a.$$

In other words, the function $a(\cdot)$ assigns to each $x \in X$ one and the same value a. We called such a function a constant function (see (4.14) *on p.87).*

[96] Actually, we have proved a very strong form of continuity. Generally speaking, $\delta = \delta(\varepsilon, x_0)$ is tailored not only to the given ε, but also to the particular argument x_0 at which continuity needs to be verified. We were able to find δ which does not depend on the argument x_0 but only on ε. We will learn later that this makes id_X **uniformly continuous** on its domain X. See def.10.34 (Uniform continuity of functions) on p.218.

Proposition 10.24 (Rules of arithmetic for continuous real–valued functions)**.** *Given is a metric space* (X,d)*. Let the functions*

$$f(\cdot), g(\cdot), f_1(\cdot), f_2(\cdot), f_3(\cdot), \dots, f_n(\cdot) : A \longrightarrow \mathbb{R}$$

all be continuous at $x_0 \in A \subseteq X$*. Then*

- ***a.*** *Constant functions are continuous everywhere on* A*.*
- ***b.*** *The product* $fg(\cdot) : x \mapsto f(x)g(x)$ *is continuous at* x_0*. Specifically,* $af(\cdot)x \mapsto a \cdot f(x)$ *is continuous at* x_0 *and, using* -1 *as a constant,* $-f(\cdot) : x \mapsto -f(x)$ *is continuous at* x_0*.*
- ***c.*** *The sum* $f + g(\cdot) : x \mapsto f(x) + g(x)$ *is continuous at* x_0*.*
- ***d.*** *Any linear combination* [97] $\sum\limits_{j=0}^{n} a_j f_j(\cdot) : x \mapsto \sum\limits_{j=0}^{n} a_j f_j(x)$ *is continuous in* x_0*.*

*Proof of **a**: Let* $\varepsilon > 0$*. We do not even have to look for a suitable* δ *to restrict the distance between two arguments* x *and* x_0 *because it is always true that* $|a(x) - a(x_0)| = |a - a| = 0 < \varepsilon$ *This proves **a**.*

*Proof of **b**: Let* $x_n \in A$ *for all* $n \in \mathbb{N}$ *such that* $x_n \to x_0$ *as* $n \to \infty$*. All we need to show is* $f(x_n)g(x_n) \to f(x_0)g(x_0)$*. It follows from prop.8.10 (Rules of arithmetic for limits) on p.132 that* $\lim_{n\to\infty} f(x_n)g(x_n) = f(x_0)g(x_0)$*. This proves **b**.*

*Proof of **c**: Let* $x_n \in A$ *for all* $n \in \mathbb{N}$ *such that* $x_n \to x_0$ *as* $n \to \infty$*. We must show* $f(x_n) + g(x_n) \to f(x_0) + g(x_0)$*. It follows from prop.8.10 (Rules of arithmetic for limits) on p.132 that* $\lim_{n\to\infty} f(x_n) + g(x_n) = f(x_0) + g(x_0)$*. This proves **c**.*

*proof of **d** (outline): The proof is done by (strong) induction.*

Base case: For $n = 2$ *the proof is obvious from parts **a**, **b** and **c**.*

Induction step: Write

$$\sum_{j=0}^{n+1} a_j f_j(x) \;=\; \Big(\sum_{j=0}^{n} a_j f_j(x)\Big) \;+\; a_{n+1} f_{n+1}(x) \;=\; I \;+\; II.$$

The left term "I" is continuous by the induction assumption and the entire sum $I + II$ *then is continous as the sum of two continuous functions (proved in **c**). This proves **d**.* ■

Example 10.9 (Vector spaces of continuous real functions)**.** Let (X,d) be a metric space. Then

$$\mathscr{C}(X,\mathbb{R}) \;:=\; \{f(\cdot) : f(\cdot) \text{ is a continuous real function on } X\} \tag{10.64}$$

of all real continuous functions on X is a vector space. Note that we have seen this before in example 9.11 (Vector spaces of real functions) on p.158 for the special case of $X \subseteq (\mathbb{R}, d_{|\cdot|})$.

The sup–norm

$$\|f(\cdot)\|_\infty \;=\; \sup\{|f(x)| : x \in X\}$$

(see (9.13) on p.167) is **not a real function** on all of $\mathscr{C}(X,\mathbb{R})$ because $\|f(\cdot)\|_\infty = +\infty$ for any unbounded $f(\cdot) \in \mathscr{C}(X,\mathbb{R})$. To avoid complications from dealing with infinity, we often restrict the scope to the subspace

$$\mathscr{C}_{\mathscr{B}}(X,\mathbb{R}) \;:=\; \{h : h \text{ is a bounded continuous real function on } X\}$$

[97] See def.9.6 (linear combinations) on p.160

(see prop.9.10 on p. 167) of the normed vector space $\mathscr{B}(X,\mathbb{R})$ of all bounded real functions on X. On this subspace the sup–norm truly is a real function in the sense that $\|f(\cdot)\|_\infty < \infty$. $\square$

10.2.3 Continuity of Polynomials (Understand this!)

Definition 10.33 (polynomials)**.** Anything that has to do with polynomials takes place in $\mathbb{R}$ and not on a metric space.

Let A be subset of the real numbers and let $p(\cdot) : A \to \mathbb{R}$ be a real function on A. $p(\cdot)$ is called a **polynomial**. if there is an integer $n \geqq 0$ and real numbers $a_1, a_2, \dots, a_n$ which are constant (they do not depend on x) so that $p(\cdot)$ can be written as a sum

$$p(x) \;=\; a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \;=\; \sum_{j=0}^{n} a_jx^j. \tag{10.65}$$

In other words, polynomials are linear combinations of the **monomials** $x \to x^k \quad (k \in (N)_0$. $\square$

Proposition 10.25 (All polynomials are continuous)**.**

The proof is left as exercise **??**.

Proposition 10.26 (Vector space property of polynomials)**.** *Sums and scalar products of polynomials are polynomials.*

Proof of ***a****. Additivity:*
Let

$$p_1(x) \;=\; a_0 + a_1x + a_2x^2 + \ldots + a_nx_1^n \;=\; \sum_{j=0}^{n_1} a_jx^j$$

and

$$p_2(x) \;=\; b_0 + b_1x + b_2x^2 + \ldots + b_nx_2^n \;=\; \sum_{j=0}^{n_2} b_jx^j$$

be two polynomials. We may assume that $n_1 \leqq n_2$. *Let* $a_{n_1+1} = a_{n_1+2} = \ldots = a_{n_2} = 0$. *Then* $p_1(x) = \sum\limits_{j=0}^{n_2} a_jx^j$*, hence*

$$\begin{aligned}
p_1(x) + p_1(x) \;&=\; \sum_{j=0}^{n_2} a_jx^j \;+\; \sum_{j=0}^{n_2} b_jx^j \\
&=\; \sum_{j=0}^{n_2} (a_j + b_j)x^j \\
&=\; \sum_{j=0}^{n_2} c_jx^j \qquad (c_j := a_j + b_j)
\end{aligned}$$

This proves that the function $p_1(\cdot) + p_2(\cdot)$ *is of the form* (10.65) *and we have shown that it is a polynomial. The proof for the sum of more than two polynomials is easily done by induction. See def.2.12 on p.18.*

*Proof of **b.** Scalar product:*

Let $p(x) \;=\; \sum_{j=0}^{n} a_j x^j$ *be a polynomial. Let* $\lambda \in \mathbb{R}$*. Then*

$$(\lambda p)(x) \;=\; \lambda p(x) \;=\; \lambda \sum_{j=0}^{n} a_j x^j \;=\; \sum_{j=0}^{n} \lambda a_j x^j \;=\; \sum_{j=0}^{n} c_j x^j \quad (c_j := \lambda a_j)$$

This proves that the function $\lambda p(\cdot)$ *is of the form* (10.65). ■

*Polnomials may not always be given in their **normalized form** (10.65) on p.216. Here is an example:*

$$\begin{aligned} p(x) \;&=\; a_0 x^0 (1-x)^n + a_1 x^1 (1-x)^{n-1} + a_2 x^2 (1-x)^{n-2} + \ldots + a_{n-1} x^{n-1} (1-x)^1 + a_n x^n \\ &=\; \sum_{k=0}^{n} a_k x^k (1-x)^{n-k} \end{aligned}$$

is a linear combination of monomials and hence a polynomial. All you need to do is "multiply out" the $x^k(1-x)^{n-k}$ *terms and then regroup the resulting mess. The so called **Bernstein polynomials*** [98]

$$p(x) \;=\; \sum_{k=0}^{n} \binom{n}{k} f(\frac{k}{n}) x^k (1-x)^{n-k}$$

are of that form.

Example 10.10 (Vector space of polynomials with sup–norm)**.** Let $A \subseteq \mathbb{R}$. It follows from (10.26) and (10.25) that the set

$$\{p(\cdot) : p(\cdot) \text{ is a polynomial on } A\}$$

of all polynomials on an arbitrary non–empty subset A of the real numbers is a subspace of the vector space $\mathscr{C}(A, \mathbb{R})$. (see example (10.9) on p.215. If A is not bounded, the sup–norm

$$\|f(\cdot)\|_\infty \;=\; \sup\{|f(x)| : x \in A\}$$

is **not a real function** on the set of all polynomials on A as its value may be ∞. Matter of fact, it can be shown that, if A is not bounded, then the only polynomials for which $\|p(\cdot)\|_\infty < \infty$ are the constant functions on A. □

10.2.4 Uniform Continuity

*It will be proved in theorem **??** (Uniform continuity on sequence compact spaces) on p.**??*** [99] *that continuous real functions on the compact set* $[0,1]$ *are uniformly continuous in the sense of the following definition.* [100]

[98] Here $f(\cdot)$ is a function, not necessarily continuous, on the unit interval $[0,1]$. The binomial coefficient $\binom{n}{k}$ is defined as $\frac{n!}{k!(n-k)!}$ where $0! = 1$ and $n! = 1 \cdot 2 \cdot 3 \cdots n$ for $n \in !\mathbb{N}$ (see ch.4 of [1] B/G Art of Proof)

[99] see chapter **??** (Continuous Functions and Compact Spaces) on p.**??**

[100] For the special case of $(X,d) = (\mathbb{R}, d_{|\cdot|})$ where $d_{|\cdot|}(x,y) = |y-x|$, see [1] Beck/Geoghegan, Appendix A.3, "Uniform continuity".

Definition 10.34 (Uniform continuity of functions)**.** Let (X, d_1), (Y, d_2) be metric spaces and let A be a subset of X. A function

$$f(\cdot) : A \to Y \qquad \text{is called } \textbf{uniformly continuous}$$

if for any $\varepsilon > 0$ there exists a (possibly very small) $\delta > 0$ such that

$$d_2(f(x) - f(y)) \ < \varepsilon \quad \text{for any } x, y \in A \text{ such that } d_1(x, y) \ < \ \delta. \tag{10.66}$$ □

Remark 10.21 (Uniform continuity vs. continuity)**.** Note the following:

A. Condition (10.66) for uniform continuity looks very close to the ε–δ characterization of ordinary continuity (10.53) on p.210. Can you spot the difference?

Uniform continuity is more demanding than plain continuity because, when dealing with the latter, you can ask for specific values of both ε and x_0 according to which you must find a suitable δ. In other words, for plain continuity

$$\delta \ = \ \delta(\varepsilon, x_0).$$

In the case of uniform continuity all you get is ε. You must come up with a suitable δ regardless of what arguments are thrown at you. To write that one in functional notation,

$$\delta \ = \ \delta(\varepsilon).$$

B. It follows that uniform continuity implies continuity but the opposite need not be true. □

Example 10.11 (Uniform continuity of the identity mapping)**.** Let us have another look at proposition(10.23) where we proved the continuity of the identity mapping on a metric space. We chose $\delta = \varepsilon$ no matter what value of x we were dealing with and it follows that the identity mapping is always uniformly continuous. □

Remark 10.22. Now that you have learned the definitions for both continuity and uniform continuity, have another look at example 3.30, p.55 in ch.3.6.3 (Quantifiers for Statement Functions of more than Two Variables) where it was explained how you could obtain one definition from the other just by switching around a $\forall$ quantifier and a $\exists$ quantifier. □

10.2.5 Continuity of Linear Functions (Understand this!)

Lemma 10.1. *Let $f : (V, \| \cdot \|) \to (W, \,|\!\cdot\!|\,)$ be a linear function between two normed vector spaces. Let*

$$\begin{aligned} a \ &:= \ \sup\{ \,\big|\, f(x) \,\big|\, \ : x \in V, \|x\| = 1\}, \\ b \ &:= \ \sup\{ \,\big|\, f(x) \,\big|\, \ : x \in V, \|x\| \leqq 1\}, \\ c \ &:= \ \sup\{ \, \frac{\big|\, f(x) \,\big|}{\|x\|} : x \in V, x \neq 0\}. \end{aligned}$$

Then $a = b = c$.

Proof: We introduce the following three sets for this proof:

$$\begin{aligned} A &:= \{ \left| f(x) \right| : x \in V, \|x\| = 1 \}, \\ B &:= \{ \left| f(x) \right| : x \in V, \|x\| \leqq 1 \}, \\ C &:= \Big\{ \frac{\left| f(x) \right|}{\|x\|} : x \in V, x \neq 0 \Big\}. \end{aligned}$$

Proof that $a = b$:
It follows from $A \subseteq B$ that $a \leqq b$. On the other hand let $x \in B$ such that $x \neq 0$ (if $x = 0$ then $f(x) = 0$ certainly could not exceed a). Let $y := \|x\|^{-1}x$. Then $y \in A$ and $\|x\|^{-1} \geqq 1$, hence

$$\left| f(y) \right| = \left| f\big(x/\|x\|\big) \right| = (1/\|x\|) \left| f(x) \right| \geqq \left| f(x) \right| .$$

We conclude that the sup over the bigger set B does not exceed the sup over A, hence $a = b$.

Proof that $a = c$:
Let $x \in C$ and $y := \|x\|^{-1}x$. Then $y \in A$ and

$$\left| f(x) \right| /\|x\| = \left| f(x)/\|x\| \right| = \left| f\big(x/\|x\|\big) \right| = \left| f(y) \right| .$$

It follows that the sup over the bigger set C does not exceed the sup over A, hence $c = b$. ■

Definition 10.35 (norm of linear functions)**.** Let $f : (V, \|\cdot\|) \to (W, |\cdot|)$ be a linear function between two normed vector spaces. We denote the quantity $a = b = c$ from lemma 10.1 by $\|f\|$, i.e.,

$$\begin{aligned} \|f\| &= \sup\{ \left| f(x) \right| : x \in V, \|x\| = 1 \} \\ &= \sup\{ \left| f(x) \right| : x \in V, \|x\| \leqq 1 \} \\ &= \sup\{ \frac{\left| f(x) \right|}{\|x\|} : x \in V, x \neq 0 \}. \end{aligned} \tag{10.67}$$

$\|f\|$ is called the **norm of** f. [101]

We note that $\|f\|$ need not be finite. □

Theorem 10.15 (Continuity criterion for linear functions)**.** *Let $f : (V, \|\cdot\|) \to (W, |\cdot|)$ be a linear function between two normed vector spaces. Then the following are equivalent.*

- **A.** *f is continuous at $x = 0$,*
- **B.** *f is continuous in all points of V,*
- **C.** *f is uniformly continuous on V,*
- **D.** $\|f\| < \infty$.

Moreover, we then have

$$\left| f(x) \right| \leqq \|f\| \cdot \|x\| \quad \textit{for all } x \in V. \tag{10.68}$$

[101] Note that we use the same notation $\|\cdot\|$ for both the norm on V and the norm of the linear function f. **Do not confuse the two!**

Proof: Clearly we have ***C*** $\Rightarrow$ ***B*** $\Rightarrow$ ***A***. *We now show* ***A*** $\Rightarrow$ ***D***.

It follows from the continuity of f at 0 that there exists $\delta > 0$ such that

$$\text{(10.69)} \qquad \text{if } z \in V \text{ and } \|z\| < \delta \text{ then } \big| f(z) \big| = \big| f(z) - f(0) \big| < 1.$$

Let $x \in V$ such that $\|x\| \leqq 1$. Then $\|\delta/2 \cdot x\| \leqq \delta/2 < \delta$, hence, according to (10.69),

$$\delta/2 \cdot \big| f(x) \big| = \big| f(\delta/2 \cdot x) \big| < 1, \quad \text{hence} \quad \big| f(x) \big| < 2/\delta.$$

Because this last inequality is true for all $x \in V$ with norm bounded by 1, it follows that

$$\|f\| = \sup\{ \big| f(x) \big| : x \in V, \|x\| \leqq 1\} < 2/\delta < \infty.$$

We have proved that ***A*** $\Rightarrow$ ***D***.

We finally show ***D*** $\Rightarrow$ ***C*** *and we do this in two steps.*

First we show ***D*** $\Rightarrow$ (10.68). *The inequality trivially holds for $x = 0$ because linearity of f implies $f(0) = 0$. If $x \neq 0$ then $\|x\| > 0$ (norms are positive definite) and the inequality follows from the last characterization of $\|f\|$ in (10.67).*

Second step: Let $\varepsilon > 0$ and $\delta := \varepsilon/\|f\|$. Let $x, y \in V$ such that $\|x - y\| < \delta$. If we can prove that this implies $\big| f(x) - f(y) \big| < \varepsilon$, then f is indeed uniformly continuous and the proof is done. We show this as follows.

$$\big| f(x) - f(y) \big| = \big| f(x-y) \big| \overset{(10.68)}{\leqq} \|f\| \cdot \|x - y\| < \|f\| \cdot \delta = \|f\| \cdot \varepsilon/\|f\| = \varepsilon. \ \blacksquare$$

Note 10.3 ($\|f\|$ is a norm)**.** Let

$$\text{(10.70)} \quad \mathscr{C}_{lin}(V,W) := \mathscr{C}_{lin}\big((V, \|\cdot\|), (W, |\cdot|)\big) := \{f : V \to W \ : f \text{is linear and continuous} \}.$$

Then $\mathscr{C}_{lin}(V,W)$ is a vector space and

$$\text{(10.71)} \qquad f \mapsto \|f\| = \sup\{ \big| f(x) \big| : \|x\| = 1\}$$

defines a norm on $\mathscr{C}_{lin}(V,W)$. $\square$

In all of the proof that that is given now let $A := x \in V : \|x\| = 1$.

A. *Proof that $\mathscr{C}_{lin}(V,W)$ is a vector space.*

Let $f, g, \in \mathscr{C}_{lin}(V,W)$. We need to show that $f + g \in \mathscr{C}_{lin}(V,W)$, i.e., $f + g$ is both linear and continuous. Linearity is immediate. We now show continuity.

Let $x \in A$. Then

$$\text{(10.72)} \qquad \big| f(x) + g(x) \big| \leqq \big| f(x) \big| + \big| f(x) \big| \leqq \|f\| + \|g\| < \infty.$$

The first inequality holds because the norm $\big| f(x) \big|$ satisfies the triangle inequality for norms. The second follows from (10.67) on p.219, and the finiteness of $\|f\| + \|g\|$ is, according to the continuity criterion for linear functions (thm.10.15 on p.219), equivalent to the continuity of both f and g.

We still must show that if $f \in \mathscr{C}_{lin}(V, W)$ and $\lambda \in \mathbb{R}$ then $\lambda f : x \mapsto \lambda f(x) \in \mathscr{C}_{lin}(V, W)$, i.e., we must show that this function is linear and continuous. Again, linearity is immediate. To show continuity we proceed as follows.

Let $x \in A$. $\| \cdot \|$ is absolutely homogeneous. Hence

$$\left\| \lambda f(x) \right\| \; = \; \left\| |\lambda| f(x) \right\| \; = \; |\lambda| \cdot \left\| f(x) \right\| . \tag{10.73}$$

It follows from prop.8.4 (positive homogeneity of the sup) on p.127 that

$$\|\lambda f\| \; = \; \sup\{ \left\| \lambda f(x) \right\| \; : \|x\| = 1\} \; = \; \sup\{ \left\| |\lambda| f(x) \right\| \; : \|x\| = 1\} \tag{10.74}$$

$$= |\lambda| \cdot \sup\{ \left\| f(x) \right\| \; : \|x\| = 1\} \; = \; |\lambda| \cdot \|f\| \; < \; \infty. \tag{10.75}$$

This proves that λf is continuous.

***B.** Proof that $\|f\|$ is a norm on $\mathscr{C}_{lin}(V, W)$.*
Because (10.72) is valid for all $x \in A$, we obtain

$$\|f + g\| \; = \; \sup\{ \left\| f(x) + g(x) \right\| \; : x \in A\} \; \leqq \; \|f\| + \|g\|. \tag{10.76}$$

This proves the triangle inequality.

Likewise, we obtain from the validity of (10.74) for all $x \in A$,

$$\|\lambda f\| \; = \; \sup\{ \left\| |\lambda| f(x) \right\| \; : x \in A\} \; = \; |\lambda| \sup\{ \left\| f(x) \right\| \; : x \in A\} \; = \; |\lambda| \|f\|. \tag{10.77}$$

This proves absolute homogeneity.

Finally we show positive definiteness. Clearly $\|f\|$ is non–negative as the sup of non–negative numbers $\left\| f(x) \right\|$. Assume that $\|f\| > 0$. Then $\delta := \frac{1}{2}\|f\| > 0$ and there exists $x_0 \in A$ such that

$$\sup\{ \left\| f(x) \right\| \; : x \in A\} - \left\| f(x_0) \right\| \; < \delta, \; \text{ i.e., } \; \|f\| - \left\| f(x_0) \right\| < \delta, \; \text{ hence } \; \left\| f(x_0) \right\| \; > \delta. \tag{10.78}$$

Positive definiteness of $\| \cdot \|$ implies that $f(x_0) \neq 0$ and hence $f \neq 0$. We have proved positive definiteness of $\| \cdot \|$. ■

10.3 Function Sequences and Infinite Series

10.3.1 Convergence of Function Sequences (Study this!)

Notation Alert:

> *This chapter makes heavy use of the notation $f(\cdot)$ instead of f for a function $X \to \mathbb{R}$ to emphasize when sequences of functions $f_n(\cdot)$ are used and when function values (real numbers) $f_n(x)$ are used.*

Vectors are more complicated than numbers because an n–dimensional vector $v \in \mathbb{R}^n$ represents a grouping of a finite number n of real numbers. Matter of fact, any such vector $(x_1, x_2, x_3, \cdots, x_n)$ can be interpreted as a real function (remember: a real function is one which maps it arguments into $\mathbb{R}$)

$$f(\cdot) : \{1, 2, 3, \cdots, n\} \to \mathbb{R} \qquad j \mapsto x_j \tag{10.79}$$

(see (9.4) on p.152).

Next come sequences $(x_j)_{j\in\mathbb{N}}$ *which can be interpreted as real functions*

$$g(\cdot) : \mathbb{N} \to \mathbb{R} \qquad j \mapsto x_j. \tag{10.80}$$

Finally we deal with any kind of real function

$$h(\cdot) : X \to \mathbb{R} \qquad x \mapsto h(x) \tag{10.81}$$

as the most general case.

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