Math 330 - Number Systems - Sample Exam Problems

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The following list of problems is typical for what you might find on my first Math 330 exam. I plan to add to this list in the future and also include sample problems for exam 2 and the final exam.

Problem 0.1. (**Induction**). Let $x_1 = 1, x_2 = 1 + \frac{1}{2}, \dots, x_k = \sum_{j=1}^{k} \frac{1}{j}$ $(k \in \mathbb{N})$.
Prove by induction that $\sum_{k=1}^{n} x_k = (n+1)x_n - n$ $(n \in \mathbb{N})$.

Problem 0.2. (**Induction**). Prove by induction that $\sum_{j=1}^{n} j(j!) = (n+1)! - 1$ $(n \in \mathbb{N})$.

Problem 0.3. (**Strong Induction**). Let $x_0 = 1,\ x_1 = 2,\ x_2 = 3,\ \dots,\ x_n = x_{n-1} + x_{n-2} + x_{n-3}$ $(n \in \mathbb{N}, n \geqq 3)$.
Prove by strong induction that $x_n \leqq 3^n$ for all $n \in \mathbb{Z}_{\geqq 0}$.

Problem 0.4. (**Strong Induction**).

Let $x_0 = 2,\ x_1 = 4,\ x_{n+1} = 3x_n - 2x_{n-1}$ for $n \in \mathbb{N}$. Prove by strong induction that $x_n = 2^{n+1}$ for every integer $n \geq 0$. Hint: Is one number enough for the base case?

Problem 0.5. (**Strong Induction**).

Let $x_0 = 1,\ x_1 = 3,\ x_{n+1} = 2x_n + 3x_{n-1}$ for $n \in \mathbb{N}$. Prove by strong induction that $x_n = 3^n$ for every integer $n \geq 0$. Hint: Is one number enough for the base case?

Problem 0.6. (**Recursion**). Let $x_1 = 3,\ x_{n+1} = x_n + 2n + 3$ $(n \in \mathbb{N}$. Prove by induction that $x_n = n(n+2)$ $(n \in \mathbb{N})$.

Problem 0.7. (**Logic**). Given a function $f : X \to Y$, negate the following statements:

a. There exists $x \in X$ and $y \in Y$ such that $f(x) = y$,

b. For all $x \in X$ there exists $y \in Y$ such that $f(x) = y$,

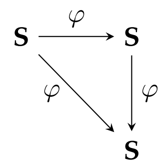
c. $\exists x \in X$ such that $\forall y \in Y$ such that $f(x) \neq y$.

d. $\forall x_1, x_2 \in X$: if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

Problem 0.8. (**Functions**). Given is a function $f : A \longrightarrow B$ $(A, B \neq \emptyset)$. Give the definitions of each of the following:

a. f is injective.
b. f is surjective.
c. f is bijective.
d. f has a left-inverse g.
e. f has a right-inverse h.

For **d** and **e**, give the "arrow diagram" which show domain and codomain for each function involved. In both cases it will like the one to the left. Each symbol **S** denotes a (possibly different) set and each symbol φ denotes a (possibly different) function.



Problem 0.9. (**Set functions**). Given is an arbitrary collection of sets $(A_j)_{j\in J}$. Determine for each assertion below whether it is true or false. If it is true, prove it. If it is false, give a counterexample.

a. $f(\bigcup_{j\in J} A_j) \subseteq \bigcup_{j\in J} f(A_j)$; **b.** $\bigcup_{j\in J} f(A_j) \subseteq f(\bigcup_{j\in J} A_j)$;

c. $f(\bigcap_{j\in J} A_j) \subseteq \bigcap_{j\in J} f(A_j)$; **d.** $\bigcap_{j\in J} f(A_j) \subseteq f(\bigcap_{j\in J} A_j)$;

You may use the fact that the direct image is increasing with its argument: $A \subseteq B \Rightarrow f(A) \subseteq f(B)$.

Problem 0.10. (**Equivalence relations and partial order relations**).

a. Let $a, b \in \mathbb{Z}$. State as precisely as possible the definition of $a \mid b$.
b. Is the relation $a|b$ **reflexive**? **symmetric**? **antisymmetric**? **transitive**? If true, prove it. If false, give a counterexample.

Problem 0.11. (**Functions and equivalence relations**).
Let $f : X \to Y (X, Y \neq \emptyset)$. Prove that $a \sim b \Leftrightarrow f(a) = f(b)$ is an equivalence relation on X.

Problem 0.12. (**Continuity**). Let $a, b, c, d \in \mathbb{R}$ such that $a < b$ and $c < d$. Let $f :]a, b[\to]c, d[$ be bijective and strictly monotone, i.e., strictly increasing or decreasing. Prove that both f and f^{-1} are continuous.

Hint: Use ε–δ continuity.

Problem 0.13. Let x_n, $\hat{x}_n$ be two convergent sequences such that $x_n \leqq \hat{x}_n$ for all $n \geqq N_1$. Let $\alpha = \lim x_n$, $\beta = \lim \hat{x}_n$. Then $\alpha \leqq \beta$.